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Exact Operating Characteristics for Linear Sum of Envelopes of Narrowband Gaussian Process and Sinewave

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Preface

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The characteristic function of a linear sum of M independent Rice variates is derived and evaluated exactly and then used in a numerical procedure to determine the exceedance distribution function, as a function of the threshold, the input signal-to-noise ratio, and M. Plots of the detection probability and false alarm probability for a wide range of signal-to-noise ratios are given, for values of M up to 8192. In addition, the required threshold values and input signal-to-ratios are tabulated and plotted for specific values of M, false alarm probability, and

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detection probability. A program and explanation are included for those users interested in extending results to their particular application.

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LIST OF SYMBOLS

M	Number of independent envelope samples summed
e_m	m-th envelope sample
x	Decision variable; sum of M envelope samples
P_D	Detection probability
P_{FA}	False alarm probability
A	Sinewave amplitude
σ	Noise standard deviation
α	A/σ, voltage measure of input signal-to-noise ratio
S/N	$\alpha^2/2$, power measure of input signal-to-noise ratio
$p_e(u)$	Probability density function of envelope random variable e_m
$f_e(\xi)$	Characteristic function of envelope random variable e_m
I_o	Modified Bessel function of order zero
${}_1F_1$	Confluent hypergeometric function
μ_x	Mean of random variable x
$f_x(\xi)$	Characteristic function of decision variable x
$Q_x(u)$	Exceedance distribution function of random variable x; (8)
Im, Re	Imaginary part, Real part
Δ	Sampling increment in ξ; (12)
N	Size of FFT; (13)-(14)
L	Limit employed on integral on ξ in (9)
b	Bias employed to shift random variable x
Φ	Cumulative distribution function of normalized Gaussian random variable; (19)
$\tilde{\Phi}$	Inverse function to Φ defined in (19)
σ_x	Standard deviation of random variable x
dB	Required input signal-to-noise ratio per-sample in decibels
β	Parameter incorporating specified P_D and P_{FA}; (28)

EXACT OPERATING CHARACTERISTICS FOR LINEAR SUM OF ENVELOPES OF NARROWBAND GAUSSIAN PROCESS AND SINEWAVE

INTRODUCTION

The operating characteristics for a linear envelope-detector of a sinewave in narrowband Gaussian noise, followed by summation of M independent envelope samples, were presented in [1] and [2, sect. 8.3]. That approach was based upon evaluation of the first 31 moments of the envelope variate and their use in a type A Gram-Charlier series approximation, or in modified approximations involving averages over different numbers of terms in the series [1, pp. 758-9]. However, there are possible pitfalls to the above approach. First, evaluation of very low exceedance probabilities, like 10^{-10}, may be inaccurate; see [1, Fig. 1]. Second, the effect of a systematic error would be hard to detect, if present, since the method yields only an approximation to the exceedance distribution function, and not its exact value.

We will use an exact approach here, based upon evaluation of the characteristic function of the envelope detector output, from which the exceedance distribution function can be precisely evaluated numerically [3,4]. In this fashion, we avoid moment evaluations altogether; we can evaluate false alarm probabilities in the 10^{-10} range easily (with double precision computer arithmetic); and we can control truncation and aliasing errors to any desired degree; see [3] for details. The results of [4] can not be applied here because each independent envelope sample is the result of a nonlinear operation, namely a square root, applied to a sum of two squares of Gaussian random variables with non-zero means.

In the plots of detection probability vs. false alarm probability to be presented herein, both abscissa and ordinate use the same normal probability scales, regardless of the number of envelope samples M considered. This allows for easier interpolation, and is in distinction to [1], where a different false alarm probability abscissa was used for each M [1, pp. 759-62]. Also, the parameter employed here for indexing the curves is α, a voltage signal-to-noise ratio which is equal to the ratio of the sinewave amplitude to the

rms noise level, rather than the dB parameter employed in [1]. This leads to curves that are more nearly equally spaced, and therefore to easier and finer interpolation capability.

Finally, we present five figures for the required input signal-to-noise ratio per sample required to realize specified false alarm and detection probabilities, as a function of M, the number of envelope samples added. The five figures correspond to detection probability P_D=.5, .9, .95, .99, and .999 respectively, and each figure contains false alarm probabilities $P_{FA}=10^{-n}$ for n=1(1)8. This total of 40 curves greatly augments the 2 cases presented in [1, Fig. 16] and [2, Fig. 8.18].

A program for the evaluation of the input signal-to-noise ratio required for a specified set of values of M, P_{FA}, and P_D is furnished, along with an explanation of its use. In this fashion, values of M, P_{FA}, and P_D intermediate to those considered here can be easily investigated.

METHOD OF EVALUATION

Characteristic Function Details

In [3,4], a method of calculating the cumulative and exceedance distribution functions directly from a given characteristic function was presented. To utilize those results here, we need the characteristic function of summation random variable

$$x = \sum_{m=1}^{M} e_m \quad , \tag{1}$$

where e_m is the envelope of a narrowband filter output with a sinewave signal of amplitude A and Gaussian noise of power σ^2. Through proper normalization, the probability density function of envelope e_m takes the familiar Rice form

$$p_e(u) = u \exp\left(-\frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) \quad \text{for } u \geq 0 \quad , \tag{2}$$

where the single parameter

$$\alpha = \frac{A}{\sigma} \tag{3}$$

is a voltage measure of signal-to-noise ratio per envelope sample. The power measure of signal-to-noise ratio per sample is

$$\frac{S}{N} = \frac{A^2/2}{\sigma^2} = \frac{\alpha^2}{2} \quad . \tag{4}$$

The quantities in (3) and (4) will be referred to as input signal-to-noise ratios, since they are per-sample measures, prior to the summation in (1) which yields the output or decision variable x.

The characteristic function corresponding to random variable e in (2) is given by Fourier transform

$$f_e(\;) = \int_{-\infty}^{+\infty} du \, \exp(i\xi u) \, p_e(u) = \int_0^{+\infty} du \; u \exp\left(i\xi u - \frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) \quad , \tag{5}$$

and will be called the Rice characteristic function. A series expansion for (5) is developed in appendix A, and has been programmed in double precision for numerical use here. As a particular special case, for $\alpha=0$, no signal, we have the Rayleigh probability density function and characteristic function:

$$p_e^{(o)}(u) = u \exp(-u^2/2) \quad \text{for } u \geq 0 \quad ,$$

$$f_e^{(o)}(\xi) = \exp(-\xi^2/2) \left[{}_1F_1\left(-\frac{1}{2}; \frac{1}{2}; \frac{\xi^2}{2}\right) + i\left(\frac{\pi}{2}\right)^{1/2} \xi \right] . \tag{6}$$

The latter follows by use of [5, 3.896 3&4] and via manipulation of the hypergeometric function series along with Kummer's transformation [5, 9.212 1]. Formula (6) is particularly attractive numerically, since the series expansion of ${}_1F_1$ contains all positive terms except for one. It should be observed that the imaginary part of Rayleigh characteristic function $f_e^{(o)}(\xi)$ in (6) decays very rapidly with ξ; this useful feature will also be shared by the Rice characteristic function, $f_e(\xi)$, and is due to the fact that the odd part of the Rice probability density function in (2) is smooth for all u, and is in fact <u>entire</u> in u, for any α. By contrast, the even part of the Rice probability density function in (2) has a discontinuous derivative for real u, thereby leading to slow decay of the real part of $f_e(\xi)$.

The characteristic function of output variable x in (1), for statistically independent envelope samples $\{e_m\}$, is given by

$$f_x(\xi) = [f_e(\xi)]^M \quad , \tag{7}$$

in terms of the Rice characteristic function (5). This relation could be used directly to find the exceedance distribution function of x according to [3, (5)-(6)]

$$Q_x(u) = \int_u^{+\infty} dt \; p_x(t) = \frac{1}{2} + \int_{0+}^{+\infty} d\xi \; \mathrm{Im}\left\{\exp(-iu\xi) \frac{f_x(\xi)}{\pi\xi}\right\} \quad . \tag{8}$$

However, the slow decay of $\mathrm{Re}\{f_x(\xi)\}$ prompts us to use a modified version given in [6, (15)]:

$$Q_x(u) = \frac{2}{\pi} \int_{0+}^{+\infty} \frac{d\xi}{\xi} \cos(u\xi) \; \mathrm{Im}\{f_x(\xi)\} \quad \text{for} \quad u > 0 \quad . \tag{9}$$

This form is applicable to positive random variables, of which x, as given by (1) and (2), is certainly a member.

To see why form (9) is preferred over (8), we develop (7) as

$$f_x(\xi) = [f_r(\xi) + if_i(\xi)]^M = \sum_{m=0}^{M} \binom{M}{m} i^m [f_i(\xi)]^m [f_r(\xi)]^{M-m} \quad , \tag{10}$$

where $f_r(\xi)$ and $f_i(\xi)$ are the real and imaginary parts of Rice characteristic function $f_e(\xi)$. Then

$$\mathrm{Im}\{f_x(\xi)\} = \sum_{\substack{m=1 \\ m \text{ odd}}}^{M} (-1)^{\frac{m-1}{2}} \binom{M}{m} [f_i(\xi)]^m [f_r(\xi)]^{M-m} \tag{11}$$

contains $f_i(\xi)$ to at least the first power in all terms, thereby yielding a rapid decay with ξ.

Development (11) has been used to show why $\mathrm{Im}\{f_x(\xi)\}$ decays rapidly with ξ. However, when we employ (9) in a numerical evaluation, we simply take the imaginary part of the power in (7), and do not use (11) at all; (11) is an alternating series of large terms for large M.

Actual numerical evaluation of (9) proceeds as follows [3]: for the Trapezoidal rule with sampling increment Δ in ξ,

$$Q_x(u) \cong \frac{2}{\pi}\left[\frac{1}{2}\mu_x\Delta + \sum_{n=1}^{\infty} \frac{1}{n} \cos(un\Delta) \; \mathrm{Im}\{f_x(n\Delta)\}\right] \quad , \tag{12}$$

where we used $f_x(\xi) \sim 1 + i\mu_x\xi$ as $\xi \to 0$. Then, restricting the u values to a particular selection,

$$Q_x\left(\frac{2\pi m}{N\Delta}\right) = \frac{2}{\pi}\left[\frac{1}{2}\mu_x\Delta + \sum_{n=1}^{\infty}\frac{1}{n}\cos(2\pi mn/N)\ \mathrm{Im}\left\{f_x(n\Delta)\right\}\right] =$$

$$= \frac{2}{\pi}\ \mathrm{Re}\sum_{n=0}^{N-1} z_n \exp(-i2\pi mn/N) \quad , \qquad (13)$$

where collapsed sequence $\left\{z_n\right\}_0^{N-1}$ is defined as

$$z_0 = \frac{1}{2}\mu_x\Delta + \sum_{j=1}^{\infty}\frac{1}{jN}\ \mathrm{Im}\left\{f_x(jN\Delta)\right\} \quad ,$$

$$z_n = \sum_{j=0}^{\infty}\frac{1}{n+jN}\ \mathrm{Im}\left\{f_x((n+jN)\Delta)\right\} \quad \text{for} \quad 1 \leq n \leq N-1 \quad . \qquad (14)$$

Form (13) is particularly attractive since it can be accomplished via an N-point FFT. It can be shown that only the values for $0 \leq m \leq N/2$ are useful in (13); the remainder are heavily aliased and must be discarded. Thus there is a trade-off: use of only the imaginary part of $f_x(\xi)$ results in aliasing twice as coarse. However, the rapid decay of the imaginary part far outweighs the aliasing.

The summations in (12) and (14) cannot be conducted to infinity. Rather the integral on ξ in (9) is terminated at limit L, where the truncation error is guaranteed to be sufficiently small. A trial and error procedure [3] yielded the following rules which control the truncation and aliasing errors:

$$\begin{aligned} L &= \min\left(9,\ 17/\sqrt{M}\right), \\ \Delta &= .12/\sqrt{M}, \\ b &= \min\left(0,\ -M\sqrt{\pi/2} + \sqrt{M}\,6\right). \end{aligned} \qquad (15)$$

The inverse $\sqrt{M}$ dependence of L and Δ for large M can be anticipated by observing that the characteristic function of random variable x in (1) then

approaches a Gaussian function with argument proportional to $M\xi^2$. The bias (or shift) b is added to random variable x in order to yield a new random variable that remains just positive, even for large M; this allows us to take maximum advantage of the fundamental aliasing interval $(0, \pi/\Delta)$ in u in (12) and (13). The linear term (in M) of b in (15) is due to the mean of the Rayleigh variate (for α=0) which is $\sqrt{\pi/2}$; the algebraic term in $\sqrt{M}$ is due to the fact that the standard deviation of random variable x in (1) increases according to $\sqrt{M}$.

In order to use this characteristic function approach, we also need the mean of random variable x in (1). Using (2), this is given by [5, 6.631 1]

$$\mu_x = M\mu_e = M \int_0^\infty du\ u^2 \exp\left(-\frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) =$$

$$= M \left(\frac{\pi}{2}\right)^{1/2} \exp\left(-\frac{\alpha^2}{2}\right) {}_1F_1\left(\frac{3}{2};\ 1;\ \frac{\alpha^2}{2}\right) \quad . \qquad (16)$$

This non-alternating series yields accurate values for the mean.

Special Cases

For general M, the characteristic function approach described above must be used. However, for M = 1 and 2, closed form expressions for the false alarm and detection probabilities are possible. Specifically, from (1) and (2), for $u \geq 0$,

$$\left.\begin{aligned} P_{FA} &= \int_u^\infty dt\ p_e(t) = \int_u^\infty dt\ t \exp(-t^2/2) = \exp(-u^2/2) \\ P_D &= \int_u^\infty dt\ t \exp\left(-\frac{t^2+\alpha^2}{2}\right) I_0(\alpha t) = Q(\alpha,u) \end{aligned}\right\} \text{for } M = 1. \quad (17)$$

And for M = 2, the false alarm probability can be determined by convolving two Rayleigh probability density functions of the form of (6), to give, for $u \geq 0$,

$$P_{FA} = \exp(-u^2/2) + \sqrt{\pi}\, u \exp(-u^2/4) \left[\Phi\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2} \right] \quad \text{for } M = 2. \tag{18}$$

Here, Φ is the cumulative distribution function of a normalized Gaussian random variable:

$$\Phi(u) = \int_{-\infty}^{u} dt\ (2\pi)^{-1/2} \exp(-t^2/2) \quad . \tag{19}$$

The detection probability of random variable x in (1) is not available in closed form for $M > 1$.

Asymptotic Performance for Large M

For large M, decision variable x in (1) is approximately Gaussian. The mean of x was given in (16); a similar approach for the mean square of x yields the variance as

$$\sigma_x^2 = M\sigma_e^2 = M(2+\alpha^2-\mu_e^2) \quad . \tag{20}$$

The probability density function of x is then approximately

$$p_x(u) \cong \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left[- \frac{(u-\mu_x)^2}{2\sigma_x^2} \right], \tag{21}$$

with exceedance distribution function

$$Q_x(u) \cong \Phi\left(\frac{\mu_x - u}{\sigma_x}\right) = \Phi\left(\frac{M\mu_e - u}{\sqrt{M}\,\sigma_e}\right) \quad . \tag{22}$$

For input signal-to-noise ratio $S/N \approx 0$, we have $\alpha = 0$ from (4), and (22), (16), and (20) specialize to

$$P_{FA} \cong \Phi\left(\frac{M\sqrt{\pi/2} - u}{\sqrt{M}\sqrt{2 - \frac{\pi}{2}}}\right) . \tag{23}$$

On the other hand, for S/N>0, (22) yields the detection probability P_D. We now use the inverse function $\tilde{\Phi}$ to definition (19) and solve (23) and (22) according to

$$\frac{M\sqrt{\pi/2}-u}{\sqrt{M}\sqrt{2-\frac{\pi}{2}}} = \tilde{\Phi}(P_{FA}) \quad , \quad \frac{M\mu_e-u}{\sqrt{M}\,\sigma_e} = \tilde{\Phi}(P_D) \quad . \tag{24}$$

Eliminating threshold u in (24), we have

$$\sigma_e\tilde{\Phi}(P_D) = \sqrt{M}\left(\mu_e-\sqrt{\frac{\pi}{2}}\right) + \sqrt{2-\frac{\pi}{2}}\,\tilde{\Phi}(P_{FA}) \quad . \tag{25}$$

But also, for large M, the required per-sample input signal-to-noise ratio α will be small, giving

$$\mu_e = \sqrt{\frac{\pi}{2}}\,{}_1F_1\left(-\frac{1}{2};\ 1;\ -\frac{\alpha^2}{2}\right) \cong \sqrt{\frac{\pi}{2}}\,(1+\frac{\alpha^2}{4}) \quad ,$$

$$\sigma_e^2 = 2 + \alpha^2 - \mu_e^2 \cong 2 - \frac{\pi}{2} \quad . \tag{26}$$

Substituting these results in (25) and solving for α, we have the required per-sample input signal-to-noise ratio measures for large M in the alternative forms

$$\alpha \cong 2\left(\frac{4-\pi}{\pi}\right)^{1/4}\frac{\beta^{1/2}}{M^{1/4}} = 1.446\,\frac{\beta^{1/2}}{M^{1/4}} \quad ,$$

$$\frac{S}{N} = \frac{\alpha^2}{2} \cong 2\left(\frac{4-\pi}{\pi}\right)^{1/2}\frac{\beta}{M^{1/2}} = 1.045\,\frac{\beta}{M^{1/2}} \quad ,$$

$$\text{dB} = 10\log\frac{S}{N} \cong 10\log\left(2\sqrt{\frac{4-\pi}{\pi}}\right)+10\log(\beta)-5\log(M) = .193+10\log(\beta)-5\log(M), \tag{27}$$

where the single parameter

$$\beta = \tilde{\Phi}(P_D) - \tilde{\Phi}(P_{FA}) \tag{28}$$

incorporates the specified false alarm and detection probabilities. (27) displays the familiar 5 log M decibel decay for large M associated with the incoherent addition in (1); see also [2, p. 279, Ex. 8.8].

RESULTS

For a given value of M, the output variable in (1),

$$x = \sum_{m=1}^{M} e_m \quad , \tag{29}$$

will exceed threshold u with false alarm probability P_{FA} when signal-to-noise ratio α is zero. That is

$$P_{FA} = \mathrm{Prob}(x>u \mid \alpha=0;\ M). \tag{30}$$

For specified values of M and P_{FA}, this relation can be solved numerically for u; the values of normalized threshold u/M are listed in table 1 for $M=2^n$, n=0(1)13 and for $P_{FA}=10^{-n}$, n=1(1)8.

The detection probability depends on threshold u, M, and signal-to-noise ratio $\alpha(>0)$:

$$P_D = \mathrm{Prob}(x>u \mid \alpha;\ M). \tag{31}$$

For specified values of M, P_D, and u, this relation can be solved numerically for the required input signal-to-noise ratio α. When the threshold results in Table 1 are employed, the results yield the required input signal-to-noise ratio for specified false alarm probability and detection probability at a particular M. These are plotted in figures 1-5 for

$$P_D = .5,\ .9,\ .95,\ .99,\ .999, \tag{32}$$

respectively. The abscissa is $\log_2 M$, and the ordinate is in decibels, as defined in (27). The fit of (27) is very good for large M, especially for the larger P_{FA} values. These results in figures 1-5 greatly extend the one in [1, Fig. 16] and [2, Fig. 8.18].

Table 1. Normalized Thresholds Required for Specified M and P_{FA}

M \ P_{FA}	1E-1	1E-2	1E-3	1E-4
1	2.14596603	3.03485426	3.71692219	4.29193205
2	1.87154046	2.46578168	2.92459903	3.31372579
4	1.68491649	2.08494224	2.39281962	2.65432267
8	1.55592564	1.82779134	2.03544098	2.21134522
16	1.46605729	1.65246898	1.79362769	1.91266565
32	1.40314416	1.53192213	1.62866385	1.70984877
64	1.35896377	1.44846093	1.51524477	1.57104117
128	1.32787317	1.39035933	1.43673968	1.47534630
256	1.30596258	1.34974198	1.38210498	1.40896493
512	1.29050601	1.32125803	1.34392160	1.36268942
1024	1.27959472	1.30123656	1.31715039	1.33030674
2048	1.27188832	1.28713956	1.29833595	1.30758095
4096	1.26644357	1.27720181	1.28509047	1.29159844
8192	1.26259580	1.27018998	1.27575385	1.28034098

M \ P_{FA}	1E-5	1E-6	1E-7	1E-8
1	4.79852591	5.25652177	5.67769243	6.06970852
2	3.65817649	3.97074674	4.25904998	4.52806135
4	2.88639585	3.09755766	3.29282208	3.47544423
8	2.36734857	2.50933650	2.64073862	2.76376208
16	2.01795589	2.11363367	2.20209577	2.28487698
32	1.78141625	1.84629005	1.90615996	1.96210527
64	1.62006566	1.66438962	1.70520835	1.74328423
128	1.50917003	1.53967893	1.56771937	1.59383081
256	1.43244246	1.45357776	1.47297026	1.49100181
512	1.37906412	1.39378248	1.40726893	1.41979378
1024	1.34176981	1.35206123	1.36148146	1.37022180
2048	1.31562790	1.32284604	1.32944798	1.33556910
4096	1.29725887	1.30233301	1.30697131	1.31126956
8192	1.28432858	1.28790149	1.29116614	1.29419029

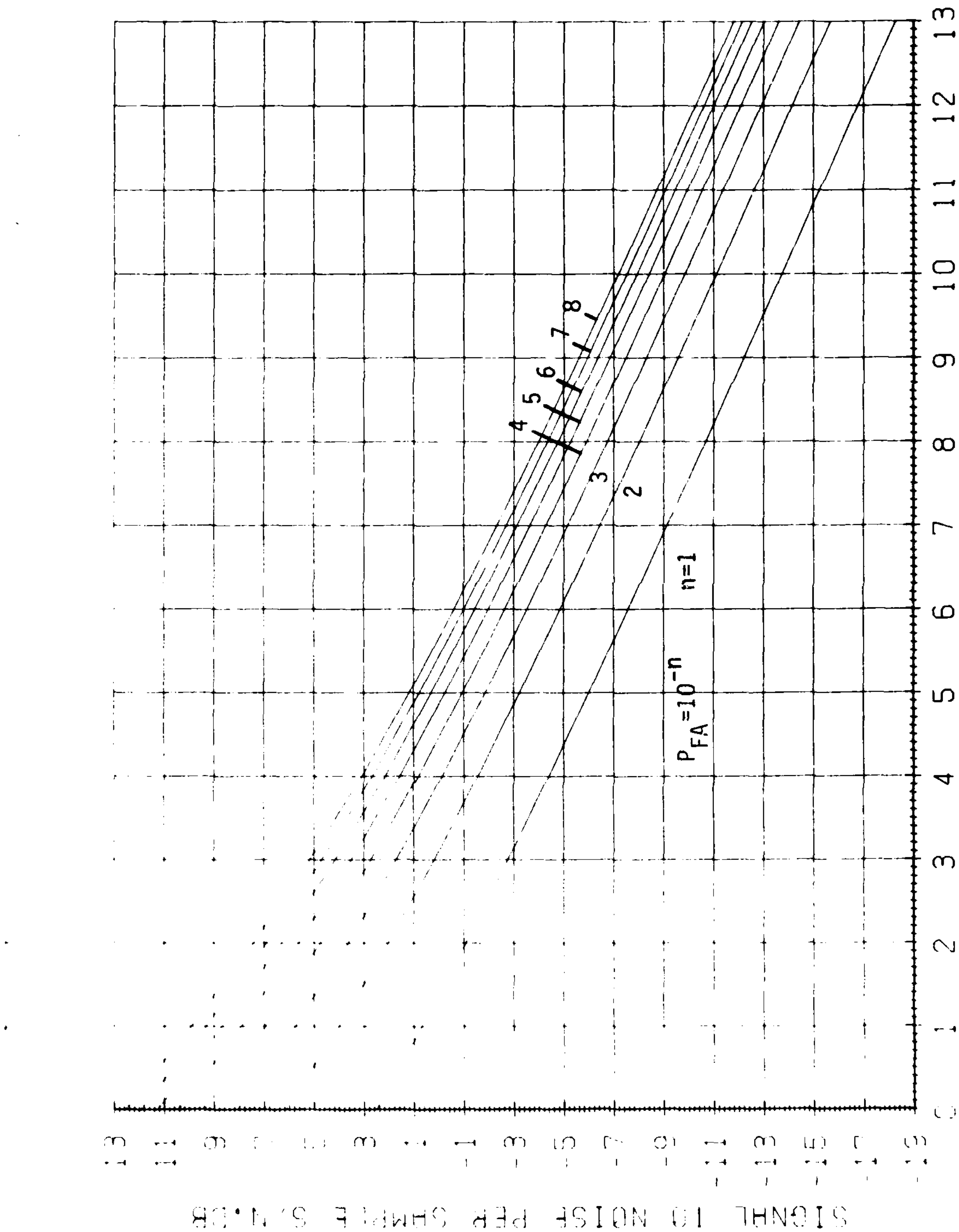

Figure 1. Required Input S/N for P_n=.5

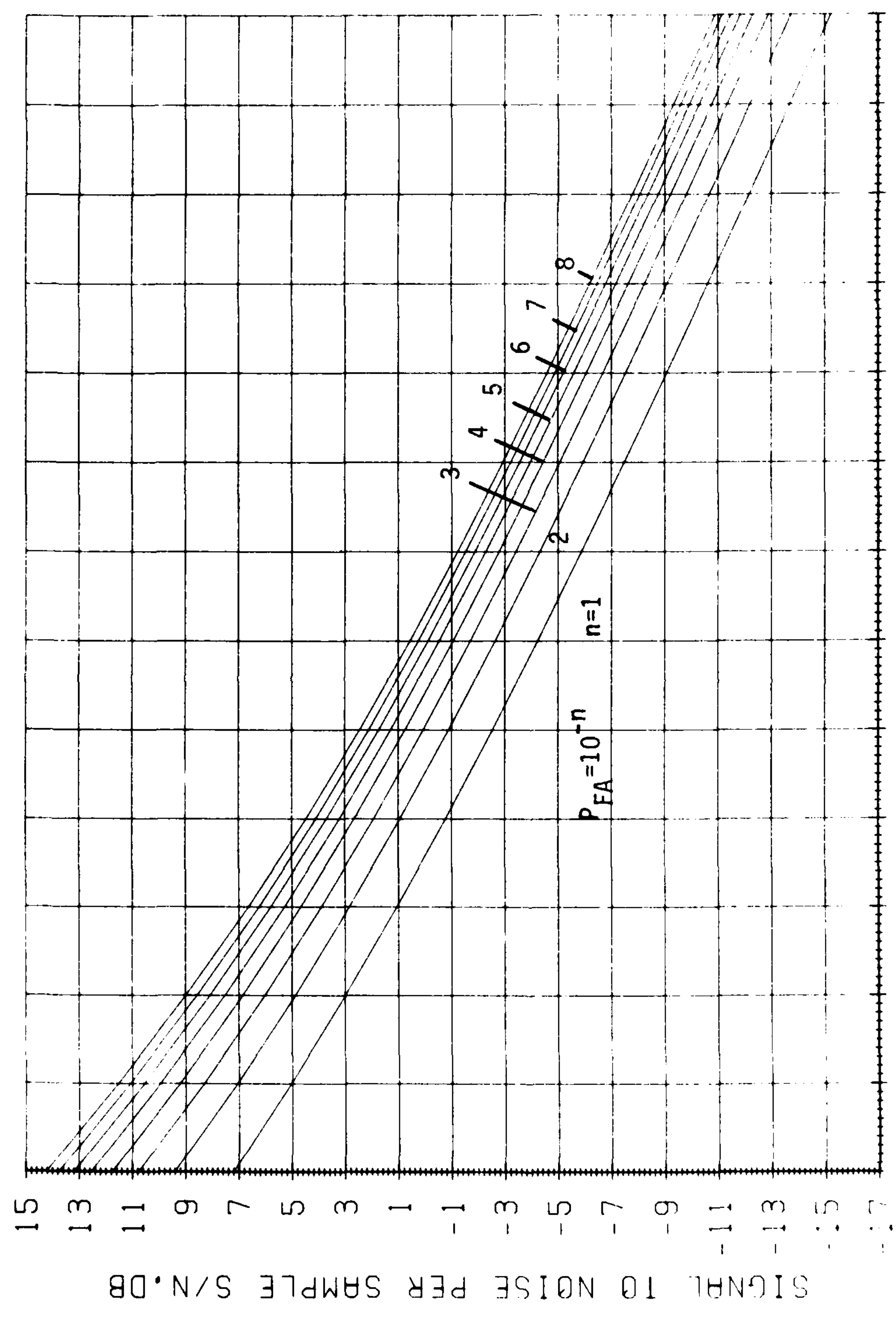

Figure 2. Required Input S/N for $P_D=.9$

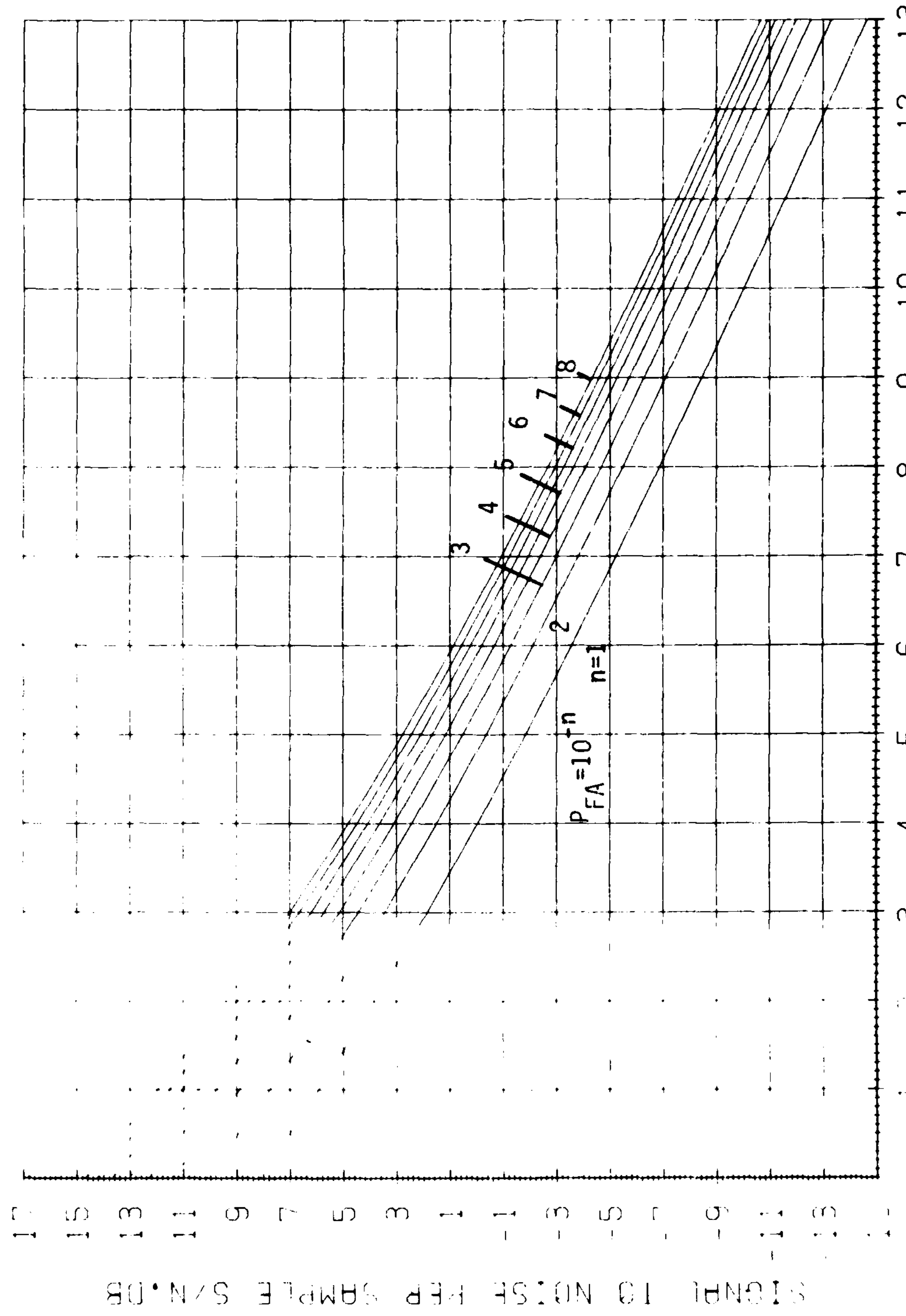

Figure 3. Required Input S/N for P_D=.95

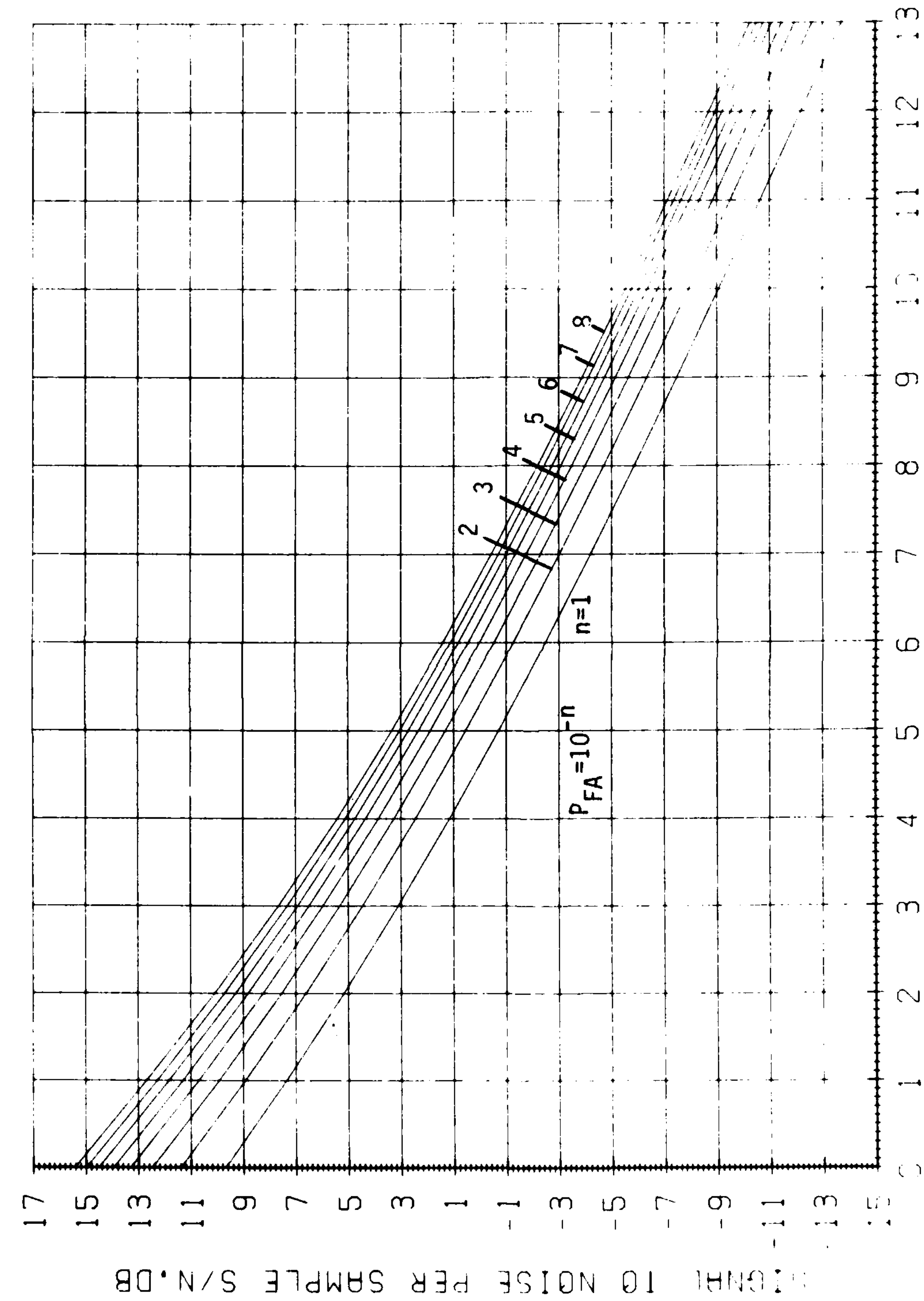

Figure 4. Required Input S/N for P_D=.99

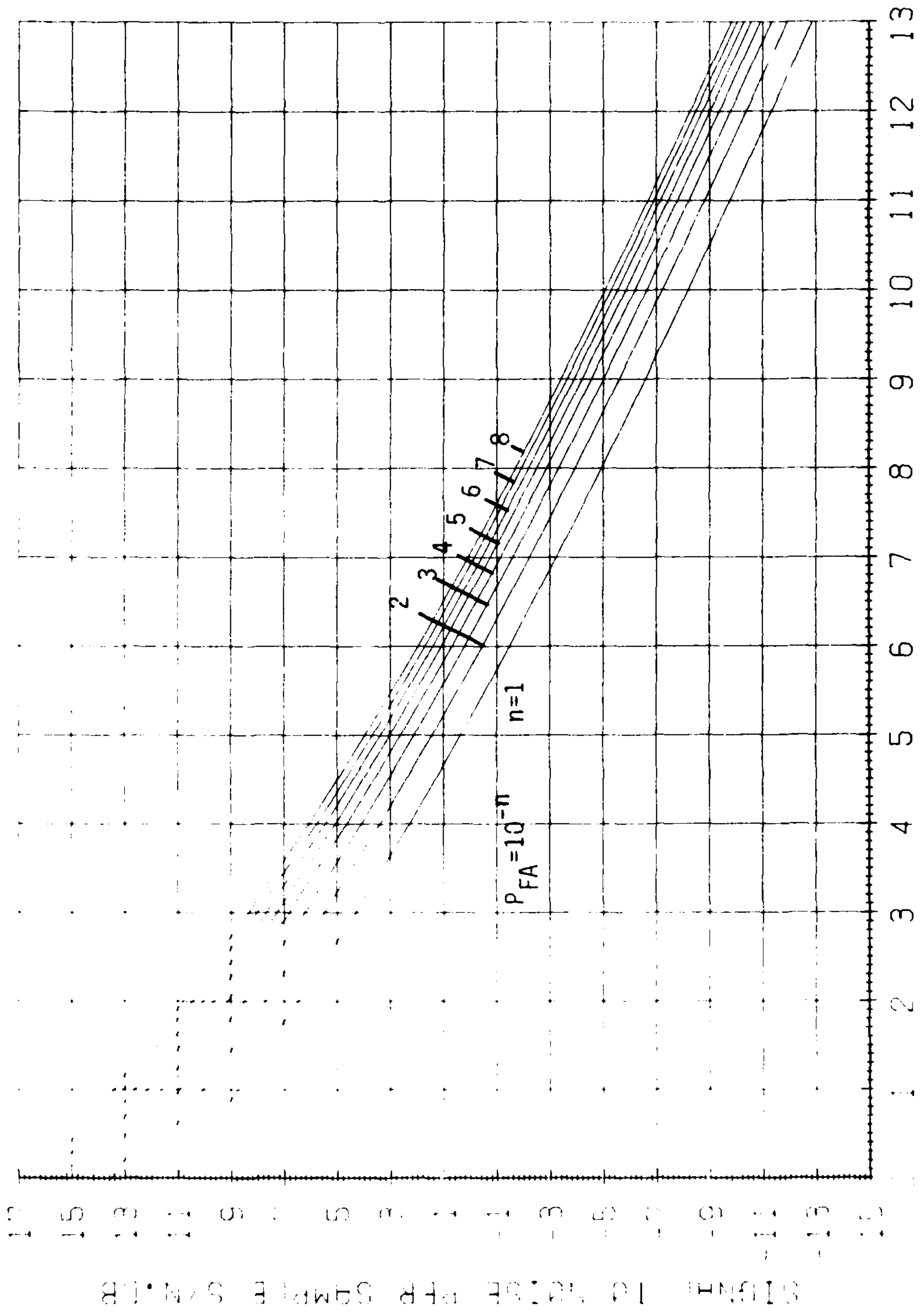

Figure 5. Required Input S/N for P_D=.999

The results in figures 1 through 5 only cover a selected set of detection and false alarm probability values. A more complete description is afforded by the receiver operating characteristics, namely detection probability vs. false alarm probability, with signal-to-noise ratio as a parameter. In figures 6 through 19 are given these operating characteristics for

$$M = 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, \quad (33)$$

respectively. The false alarm probability covers the range 10^{-10} to .5, while the detection probability covers 10^{-10} to .999. Both abscissa and ordinate in these figures employ the inverse function to the Gaussian cumulative distribution function Φ defined in (19); thus, a truly Gaussian random variable would plot as a series of equally spaced parallel straight lines (with parameter α). Observe that the curves are nearly equally spaced with parameter α, except for very small α, where the nonlinear envelope operation causes small signal suppression and a crowding together of the curves.

If the decision variable x is presumed Gaussian, and the operating characteristics overlayed on the exact results in figures 6-19, it is found that the two sets of curves for M=8192 are virtually identical in the range of P_{FA} and P_D plotted. However, for M=16, the Gaussian approximation is somewhat optimistic; for example, the exact curve for α=2.75 is well-approximated by the Gaussian approach for α=2.62. For small M, the Gaussian approximation is overly optimistic for small P_{FA}; however, the two sets cross near P_{FA}=.5, which is not a practical range of interest anyway.

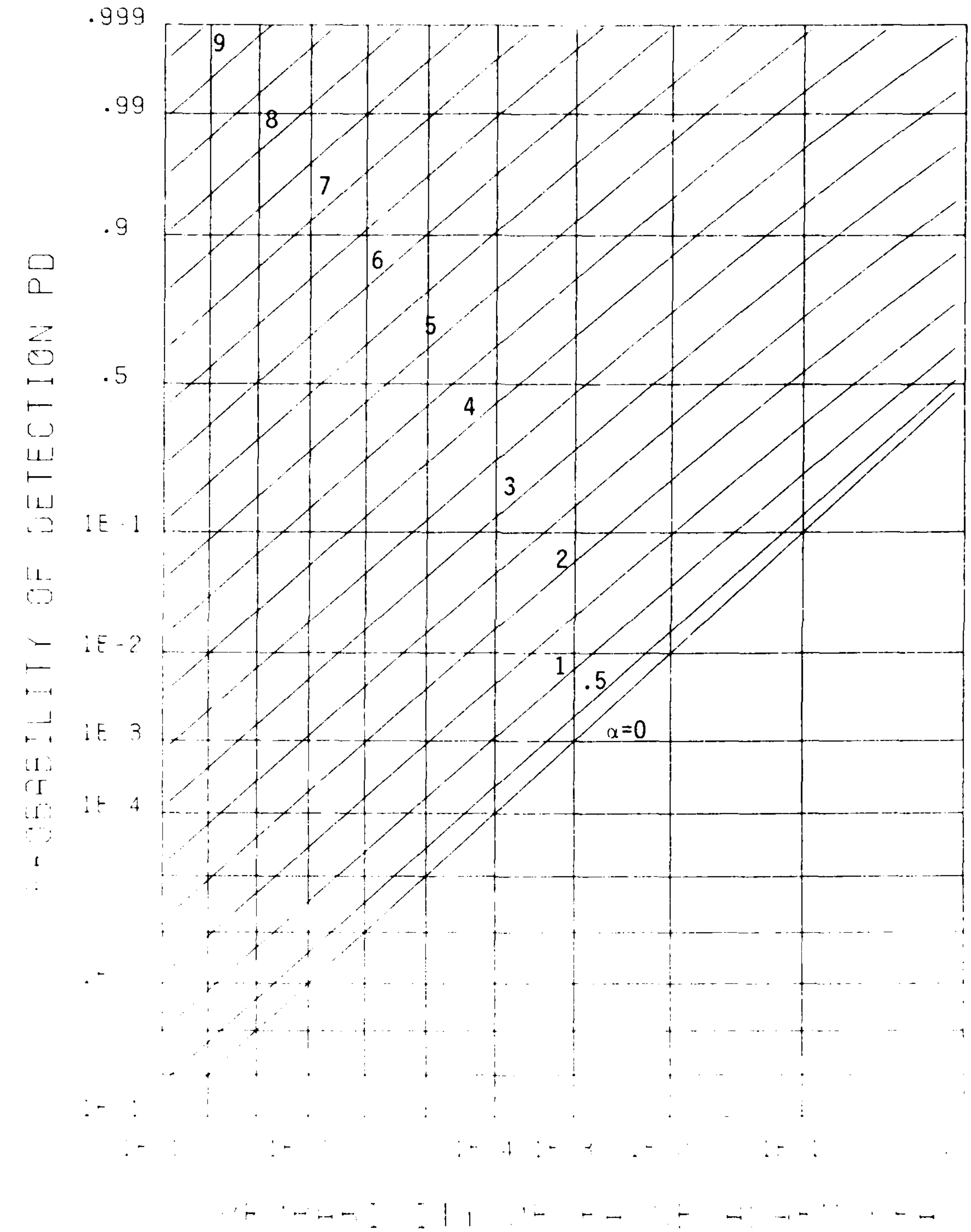

Figure 6. Receiver Operating Characteristics for M=1

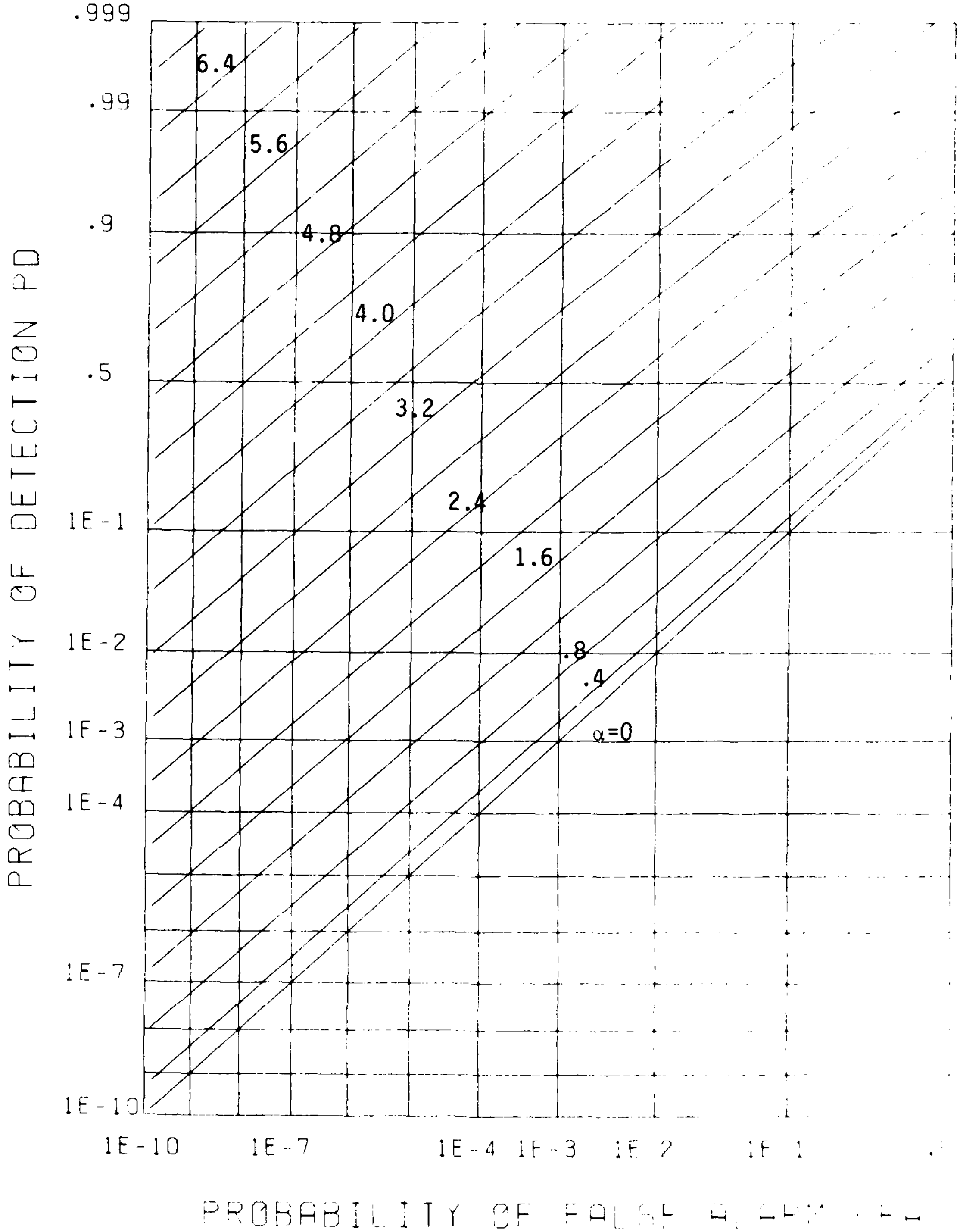

Figure 7. Receiver Operating Characteristics for M=2

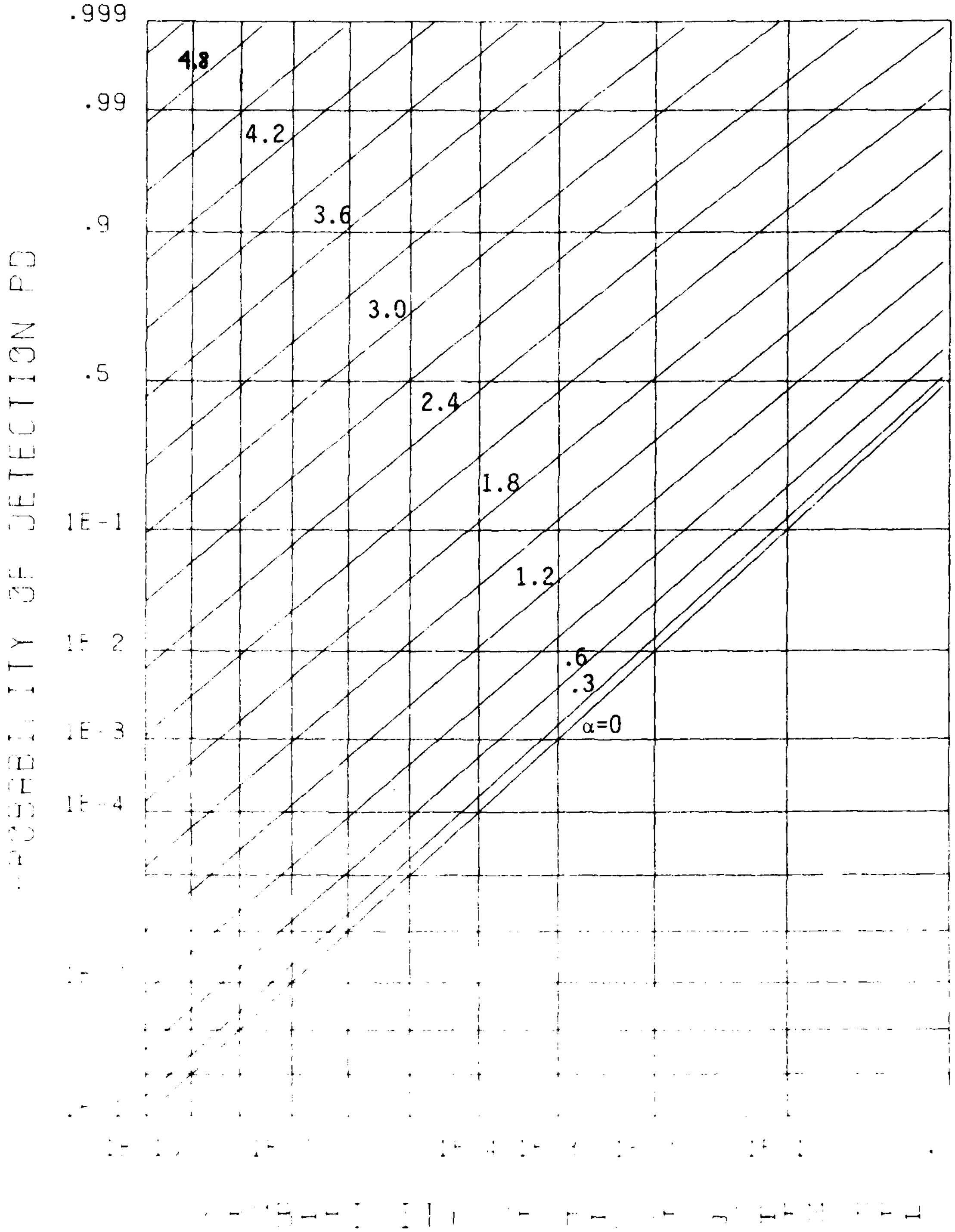

Figure 8. Receiver Operating Characteristics for M=4

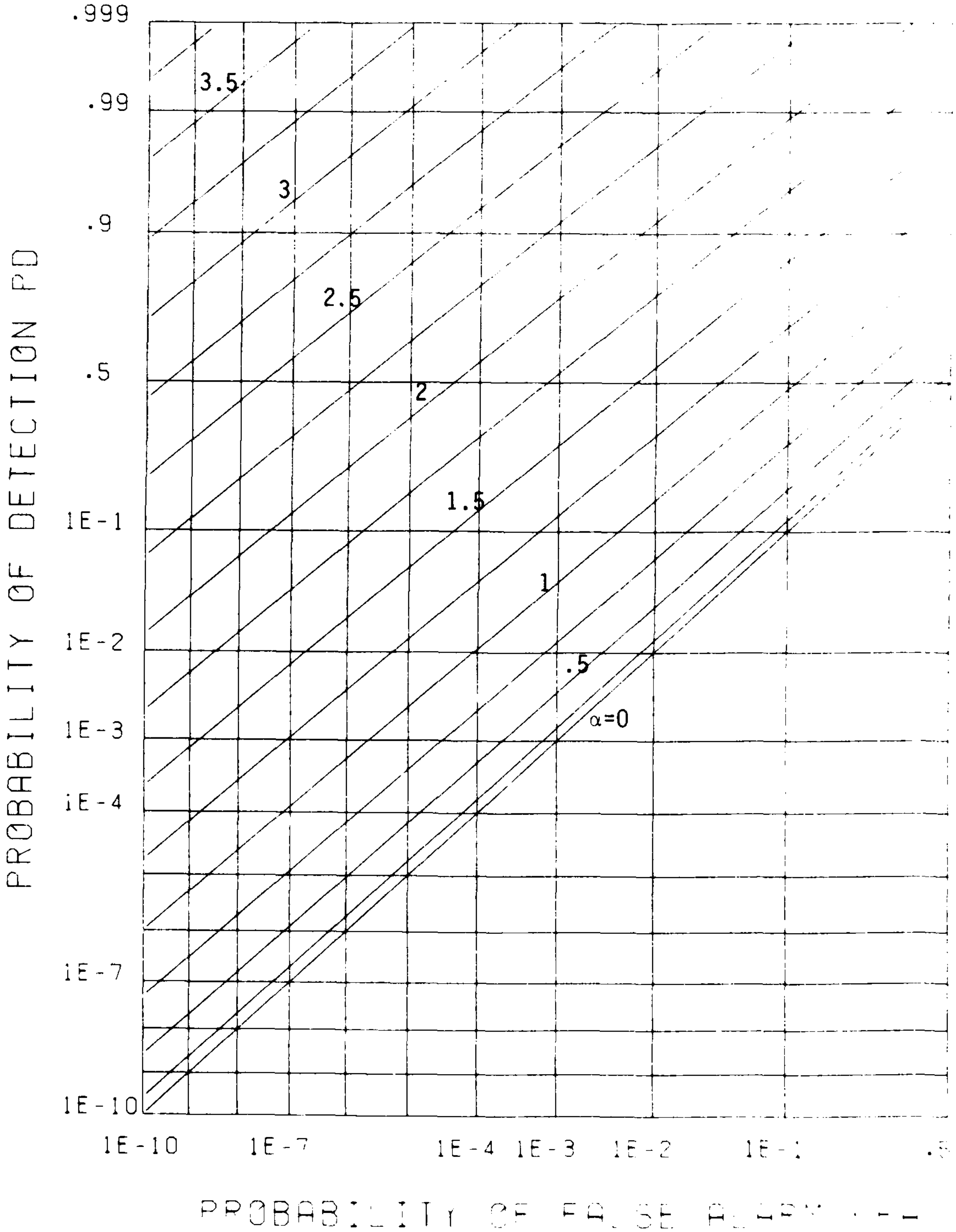

Figure 9. Receiver Operating Characteristics for M=8

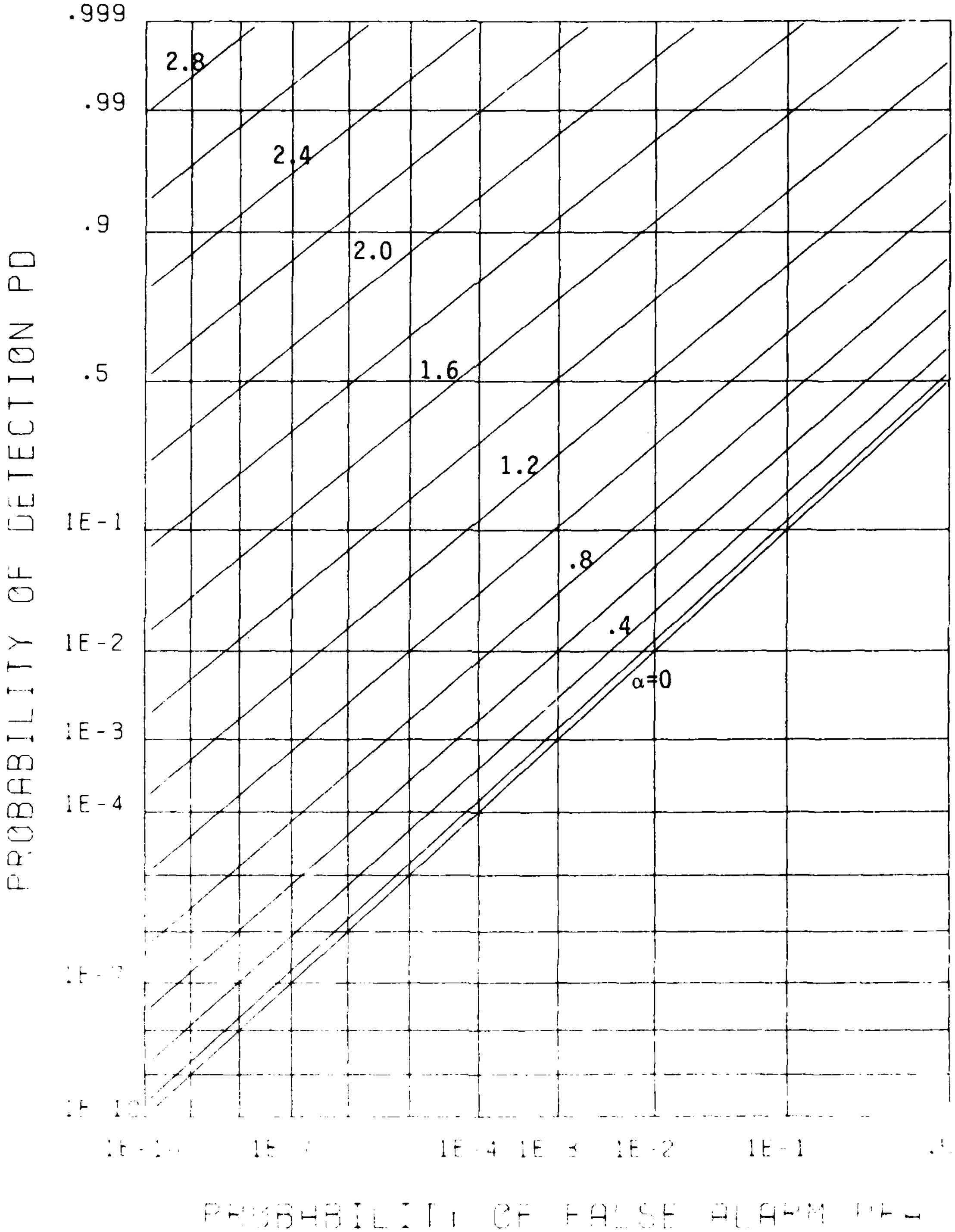

Figure 10. Receiver Operating Characteristics for M=16

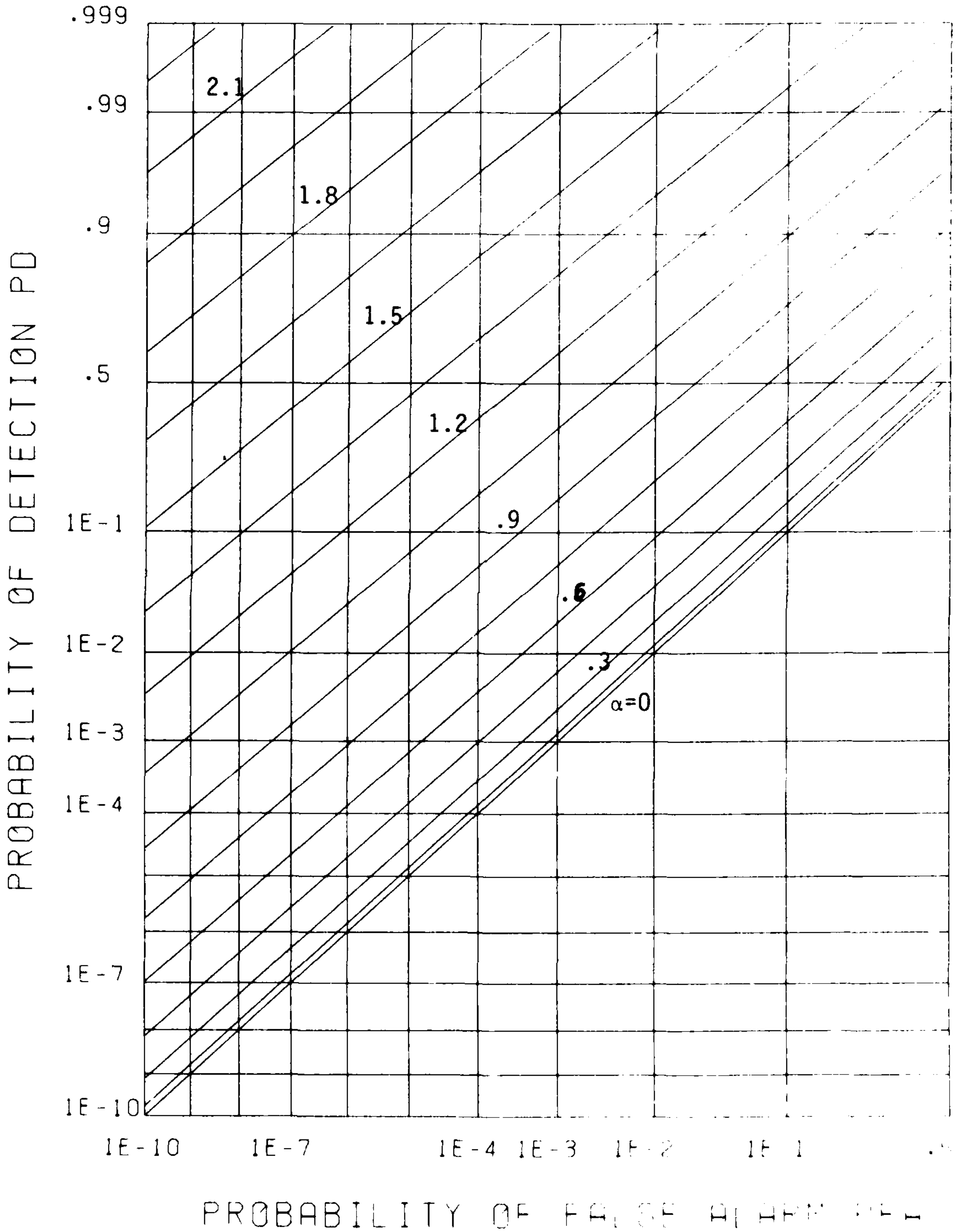

Figure 11. Receiver Operating Characteristics for M=32

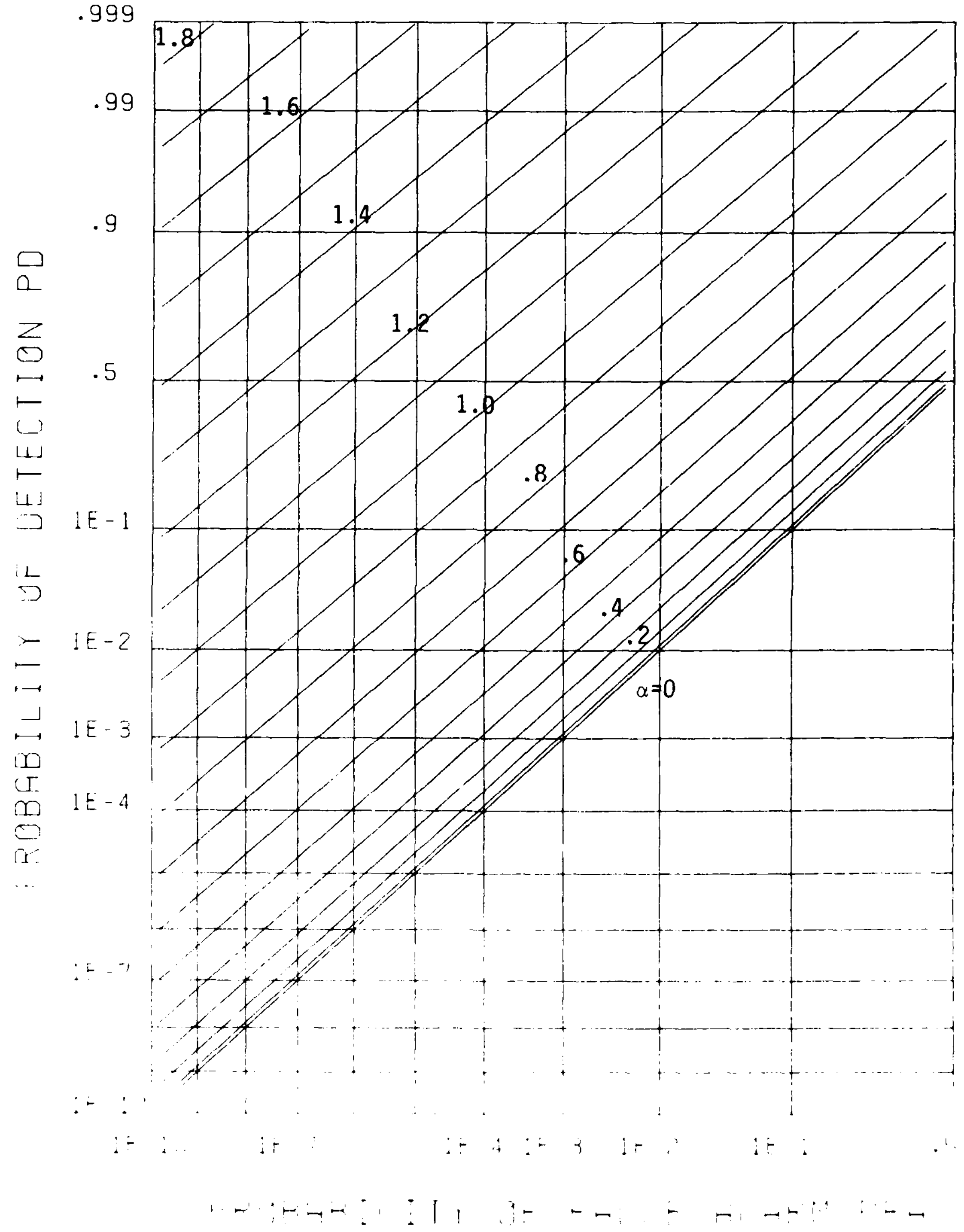

Figure 12. Receiver Operating Characteristics for M=64

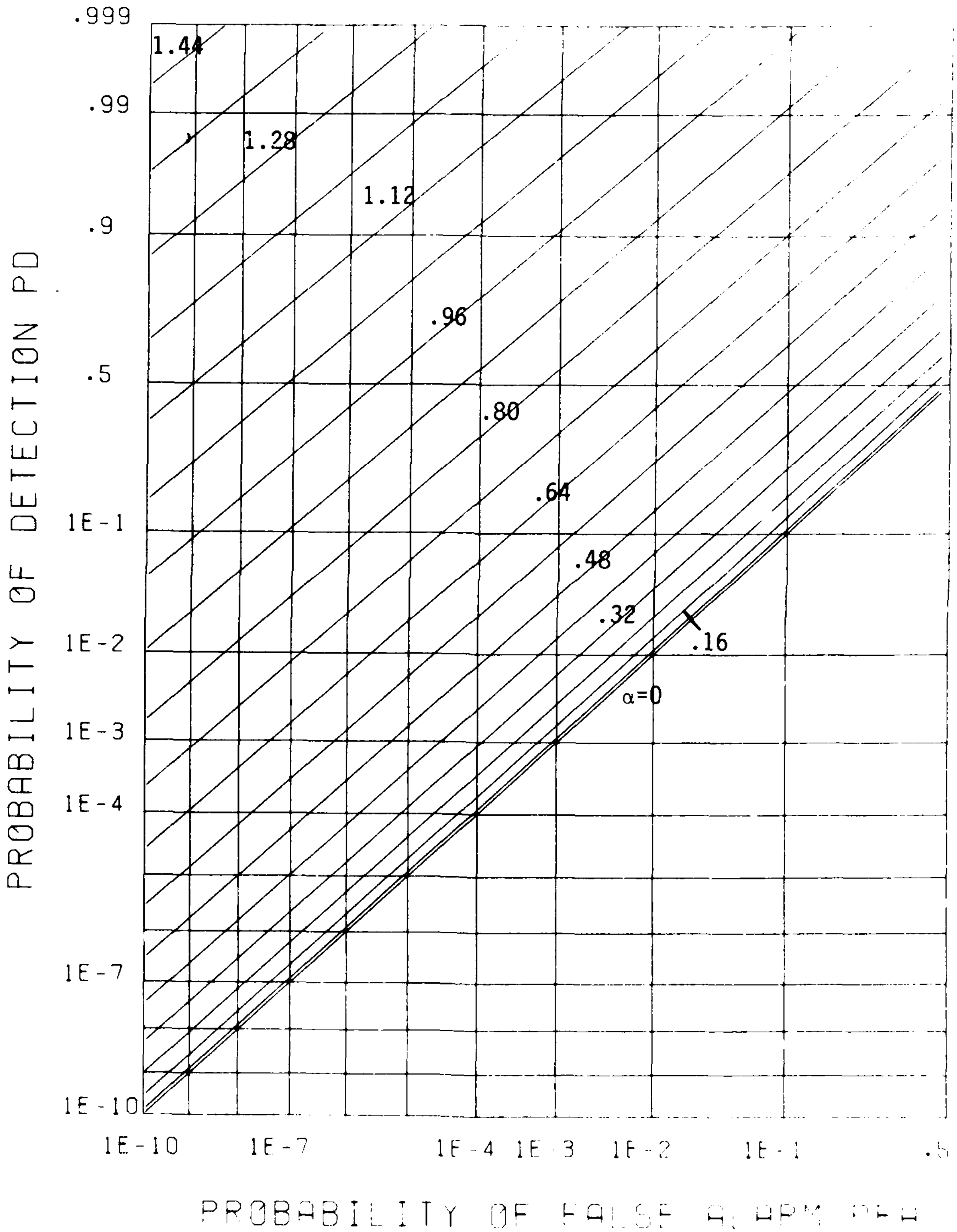

Figure 13. Receiver Operating Characteristics for M=128

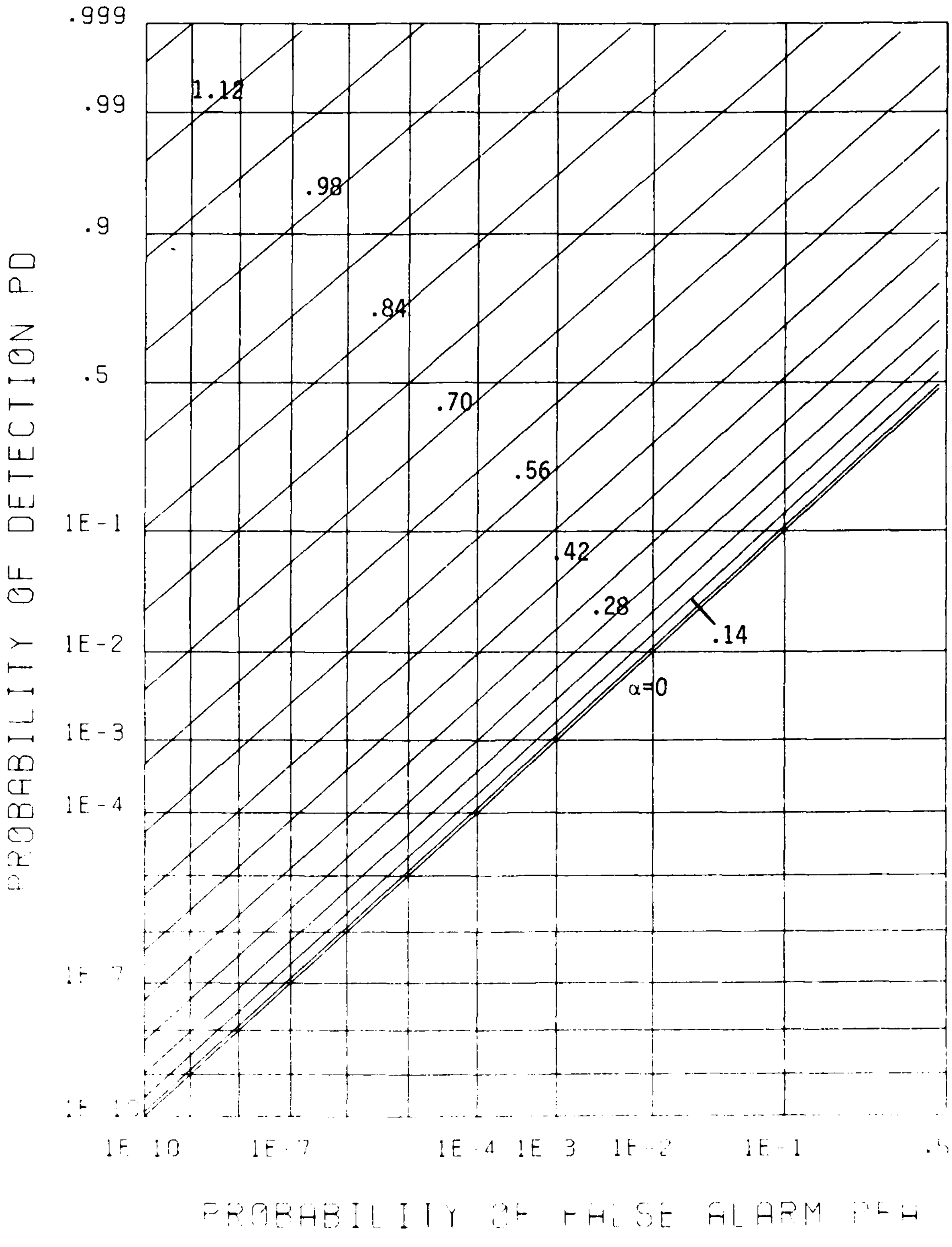

Figure 14. Receiver Operating Characteristics for M=256

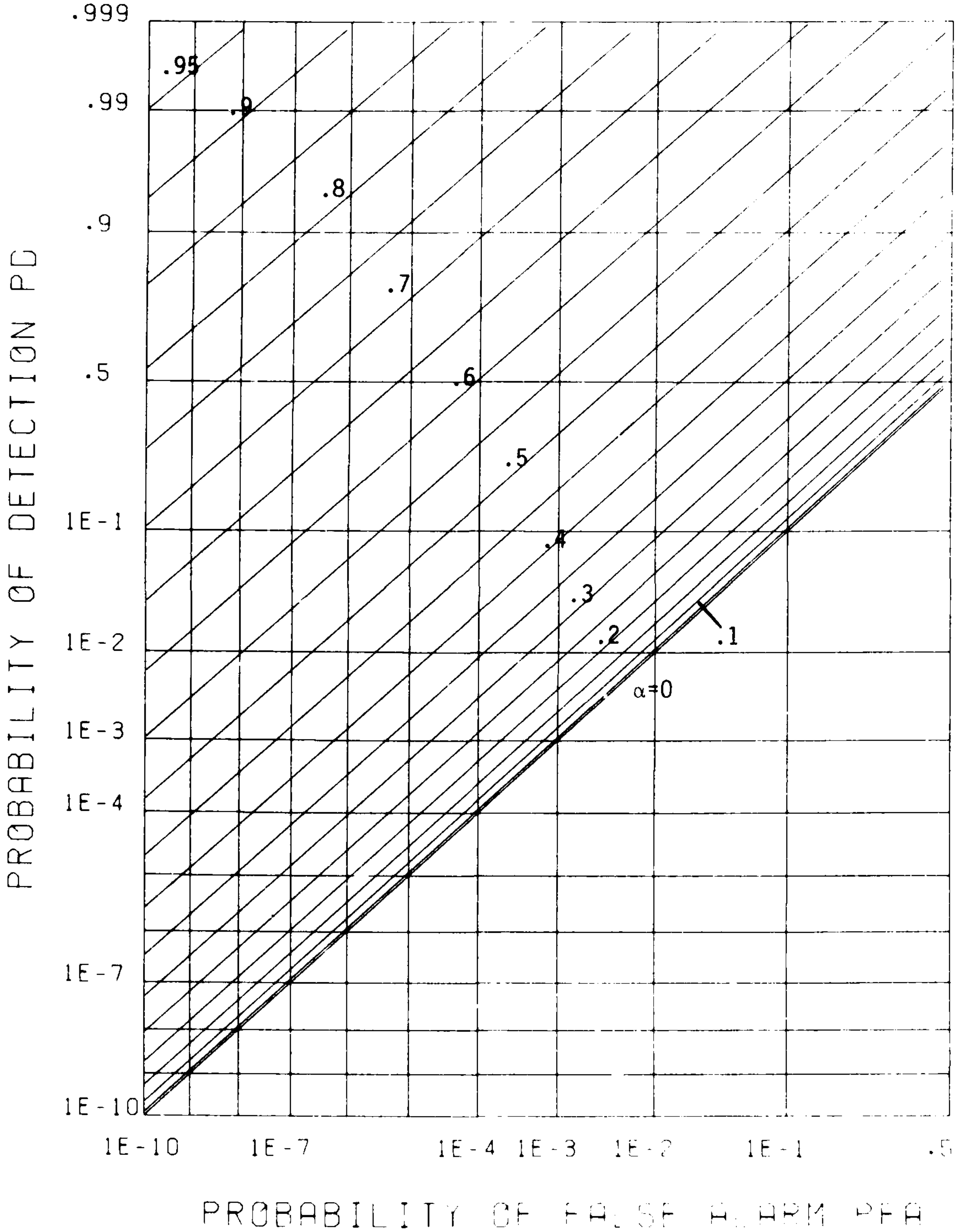

Figure 15. Receiver Operating Characteristics for M=512

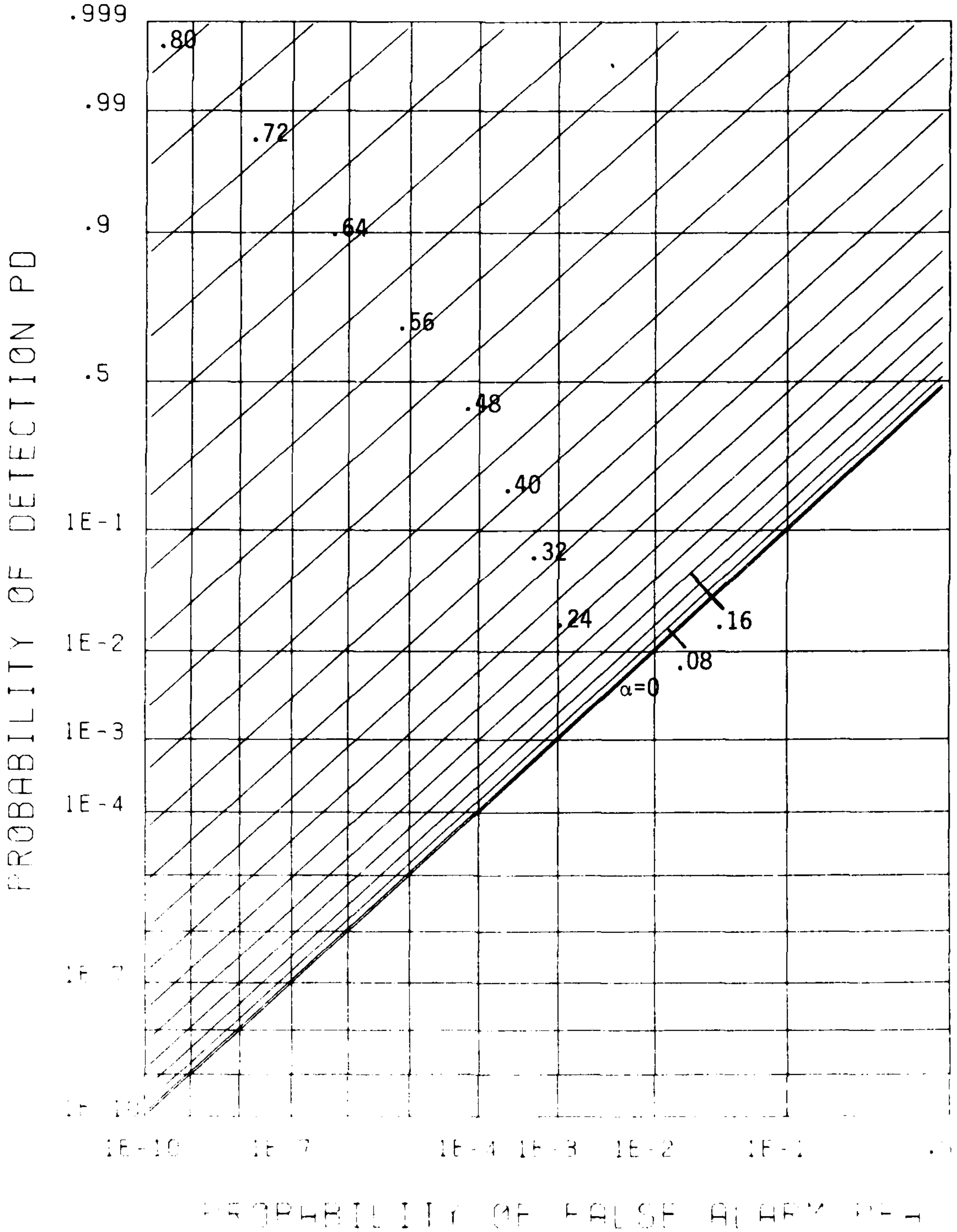

Figure 16. Receiver Operating Characteristics for M=1024

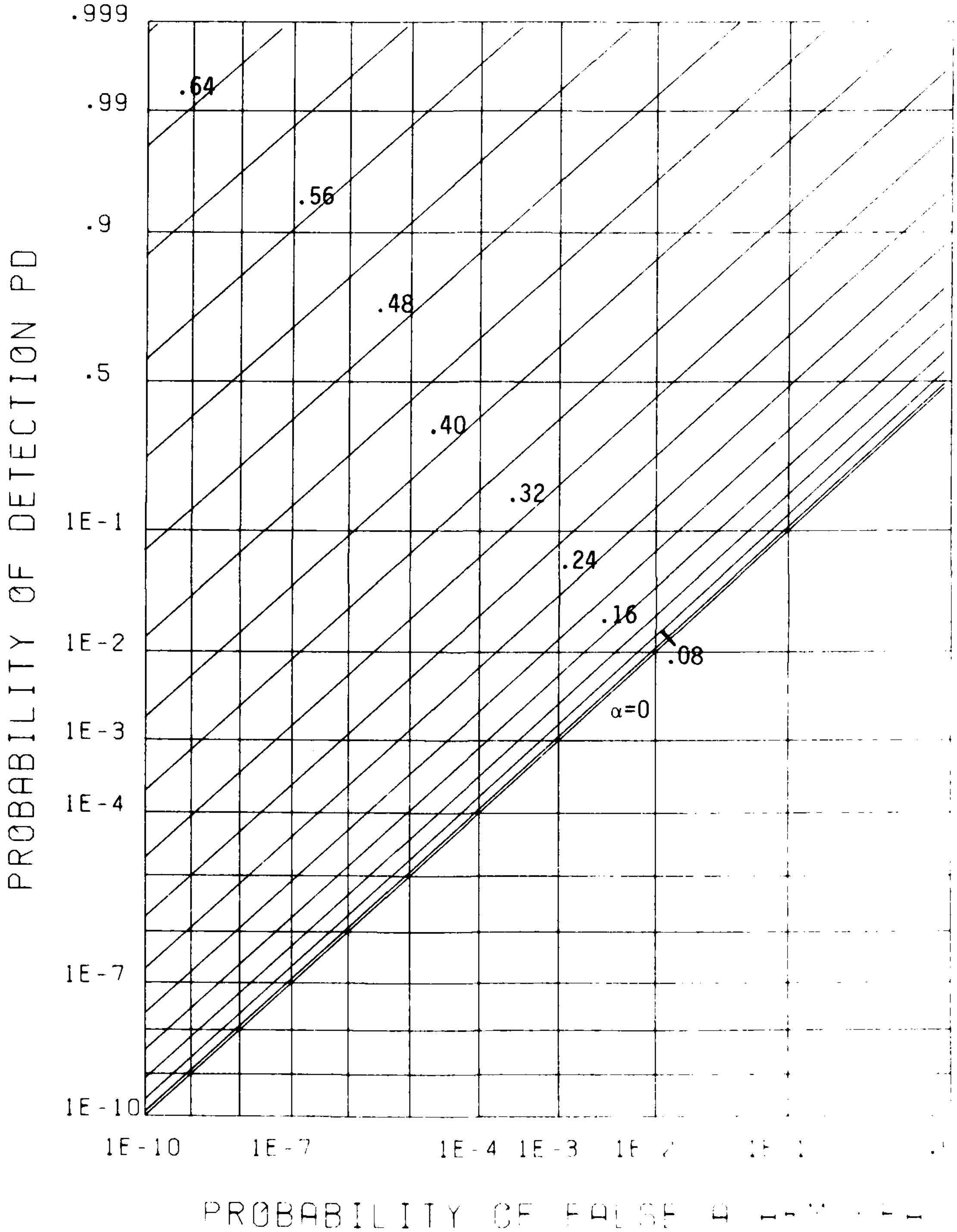

Figure 17. Receiver Operating Characteristics for M=2048

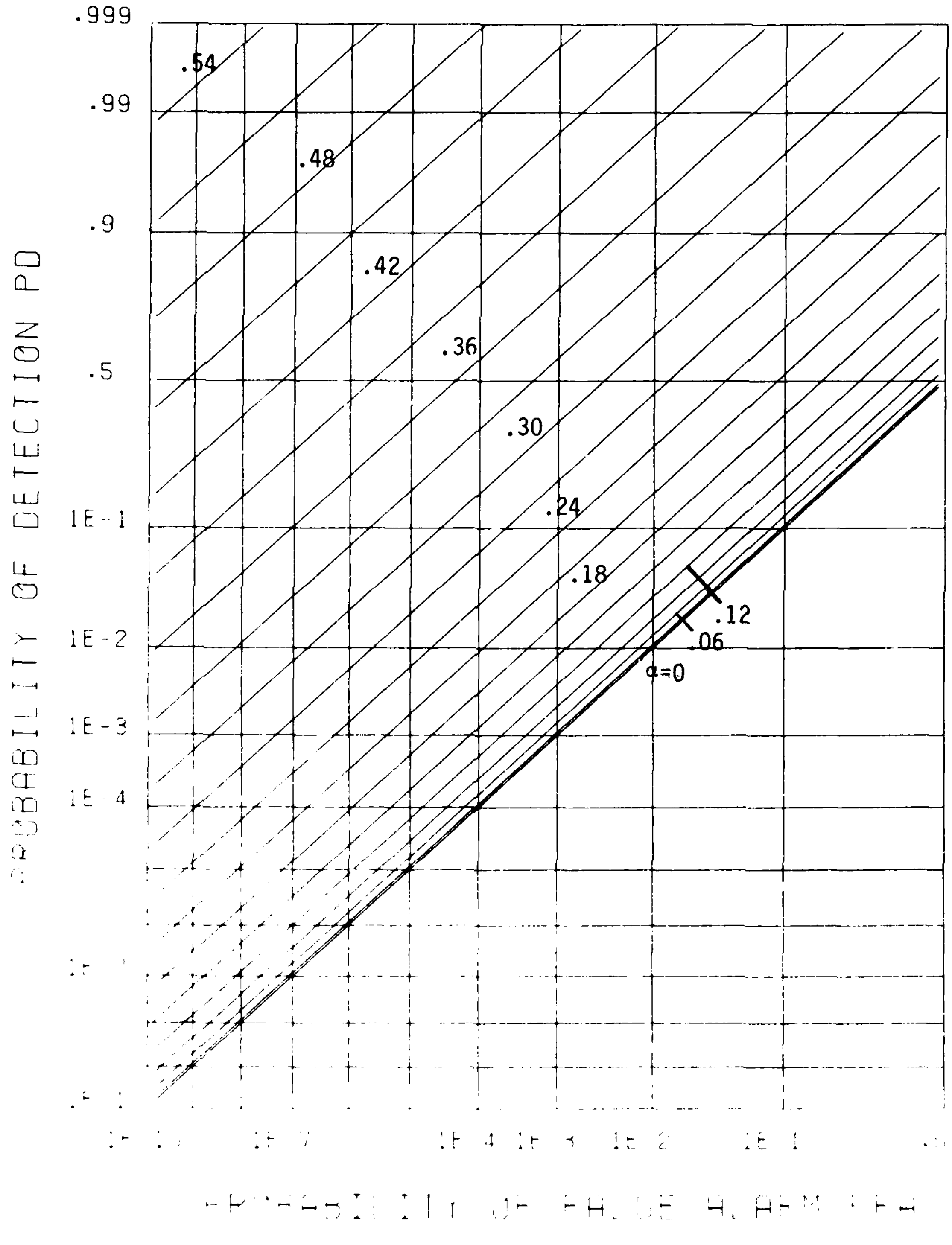

Figure 18. Receiver Operating Characteristics for M=4096

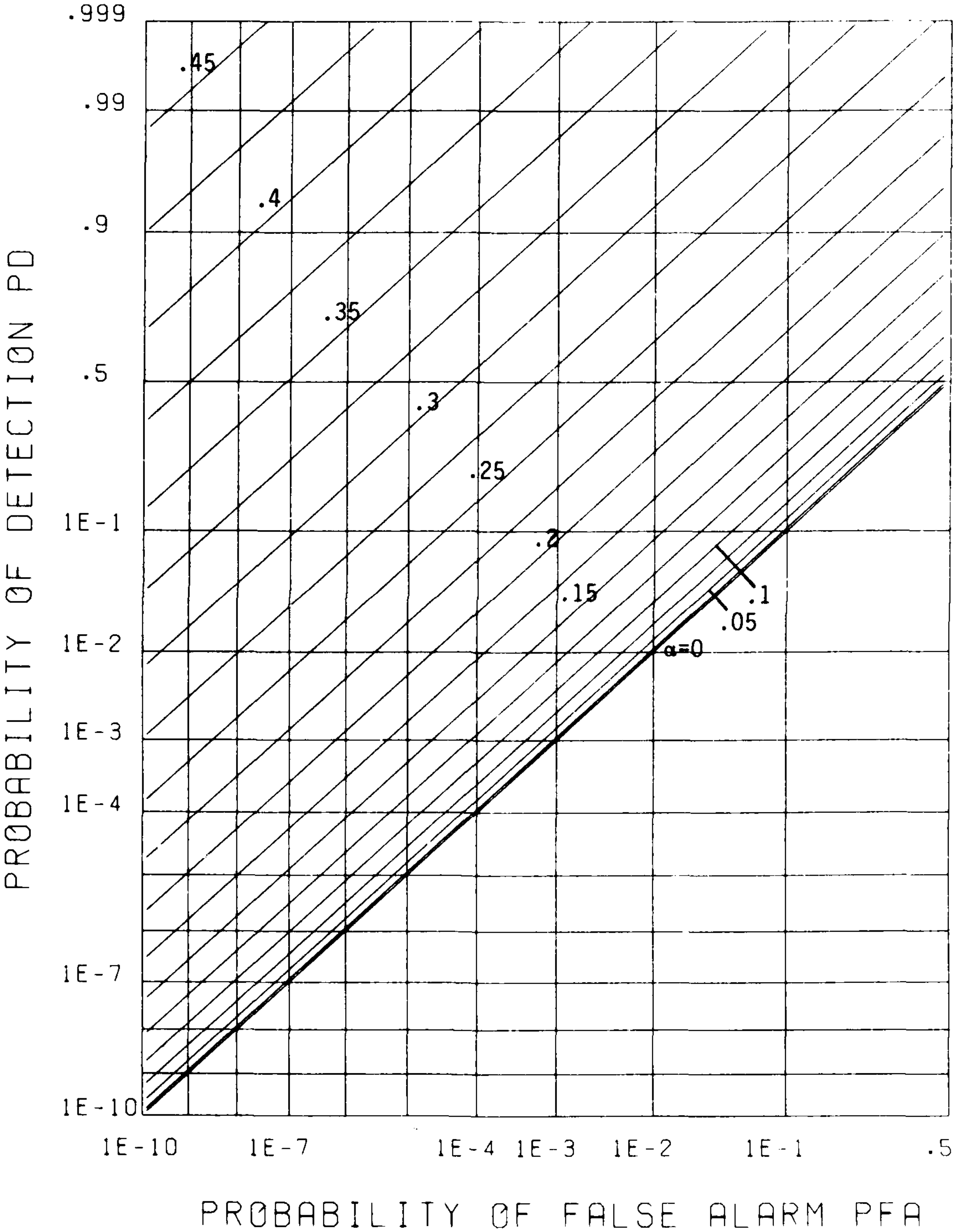

Figure 19. Receiver Operating Characteristics for M=8192

SUMMARY

A method for exact evaluation of the exceedance distribution function, of a linear sum of M envelopes of a narrowband Gaussian process and sinewave, has been utilized to determine the receiver operating characteristics for a wide range of values of M and signal-to-noise ratio. Also, the required input signal-to-noise ratio vs. M has been determined for a selected set of false alarm and detection probabilities. Programs are also supplied by which other values of the various parameters can be investigated by the user.

Agreement between the current results and those in [1,2] is very good over the range of common values plotted. For M larger than 8192, the approximation given in (27) and (28) is recommended, since the summation variable is then well represented by a Gaussian random variable.

APPENDIX A. DERIVATION OF RICE CHARACTERISTIC FUNCTION

The normalized probability density function of a Rice random variable was given in (2) as

$$p_e(u) = u \exp\left(-\frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) \quad \text{for} \quad u \geq 0 \quad . \tag{A-1}$$

The corresponding characteristic function is

$$f_e(\xi) = \int_{-\infty}^{+\infty} du \exp(i\xi u)\, p_e(u) = \int_0^{+\infty} du\, u \exp\left(i\xi u - \frac{u^2+\alpha^2}{2}\right) I_0(\alpha u) =$$

$$= \exp(-r) \sum_{n=0}^{\infty} \frac{(r/2)^n}{(n!)^2} \int_0^{\infty} du\, u^{2n+1} \exp(i\xi u - u^2/2) \quad , \tag{A-2}$$

where we have expanded I_0 in a power series [5, 8.447 1] and defined power signal-to-noise ratio

$$r = \alpha^2/2 \quad . \tag{A-3}$$

(If desired, a power series in ξ could be developed by expanding $\exp(i\xi u)$ in a power series instead of I_0.)

We define

$$C_n(\xi) = \frac{1}{2^n (n!)^2} \int_0^{\infty} du\, u^{2n+1} \exp(i\xi u - u^2/2) \quad \text{for } n \geq 0 \quad , \tag{A-4}$$

and get the characteristic function series

$$f_e(\xi) = \exp(-r) \sum_{n=0}^{\infty} r^n C_n(\xi) \quad . \tag{A-5}$$

In order to get a recurrence on $C_n(\xi)$, we also define

$$B_k(\xi) = \int_0^{\infty} dw\, w^k \exp(i\xi w - w^2/2) \quad \text{for } k \geq 0 \quad , \tag{A-6}$$

for then

$$C_n(\xi) = \frac{B_{2n+1}(\xi)}{2^n (n!)^2} \quad . \tag{A-7}$$

By integrating by parts on (A-6), there follows

$$B_k = i\xi B_{k-1} + (k-1) B_{k-2} \quad \text{for } k \geq 1 \quad . \tag{A-8}$$

This recurrence can be started with [5, 3.896 3&4]

$$B_0 = \exp(-\xi^2/2) \left[\sqrt{\frac{\pi}{2}} + i\xi \; {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{\xi^2}{2}\right) \right] \quad . \tag{A-9}$$

By looking at three adjacent terms of recurrence (A-8), we can generate the alternative recurrence

$$B_k = (2k-3-\xi^2) B_{k-2} - (k-2)(k-3) B_{k-4} \quad . \tag{A-10}$$

By means of (A-7), this translates into

$$C_n = \frac{1}{n^2} \left[\left(2n - \frac{1+\xi^2}{2}\right) C_{n-1} - \frac{n - \frac{1}{2}}{n-1} C_{n-2} \right] \quad \text{for } n \geq 2 \quad . \tag{A-11}$$

Starting values are (via manipulation of hypergeometric series and Kummer's transformation) expressable as

$$C_0 = \exp(-\xi^2/2) \left[{}_1F_1\left(-\frac{1}{2}; \frac{1}{2}; \frac{\xi^2}{2}\right) + i\sqrt{\frac{\pi}{2}}\,\xi \right] ,$$

$$C_1 = \exp(-\xi^2/2) \left[{}_1F_1\left(-\frac{3}{2}; \frac{1}{2}; \frac{\xi^2}{2}\right) + i\sqrt{\frac{\pi}{2}}\,(3-\xi^2)\frac{\xi}{2} \right] . \tag{A-12}$$

Each of the series for ${}_1F_1$ consists of terms of the same polarity, except for one term, and are therefore useful for obtaining very accurate initial values. C_0 is the characteristic function of the Rayleigh probability

density function. Relations (A-11)-(A-12) constitute recurrences on both the real and imaginary parts of C_n.

It was found that the terms $\exp(-r)\ r^n$ in (A-5) became very large for large n, while the $C_n(\xi)$ terms became very small. In order to avoid overflow and underflow, we defined the total term

$$A_n = \exp(-r)\ r^n\ C_n(\xi) \quad . \tag{A-13}$$

Reference to (A-11) readily yields the recurrence on A_n, and (A-12) furnishes corresponding obvious starting values for A_0 and A_1.

APPENDIX B. DESCRIPTION OF PROGRAMS AND LISTINGS

Overview

Information obtained via evaluation of the Rice characteristic function may be displayed in three formats.

FORMAT 1: Display PD vs. PFA

The user defines the number of samples M and the range of values for alpha, a voltage signal-to-noise ratio measure. An algorithm then utilizes the Rice characteristic function for alpha=0 and for the alphas specified by the user. This results in the production of a threshold vs. PFA and M (alpha=0) and threshold vs. PD and M (alpha>0) tables. These two tables are stored on an output file. For each user-defined M, a plot routine displays PD vs. PFA for the set of user-defined alphas.

FORMAT 2: Display SNR vs M

The user supplies the input which specifies a PD. The algorithm then solves for the threshold values corresponding to PFA=10**(-IPFA), (IPFA=1,..,8) and M=2**IM, (IM=0,...,13) and alpha=0. A root finding technique is then employed to solve for the SNR defined by a threshold value and user-defined PD. An SNR is found for each threshold value. The results are stored in an output file. A plot routine displays the required SNR vs. M for PFA=10**(-IPFA), (IPFA=1,2,...,8).

FORMAT 3: Print SNR

The user specifies a value for PD, PFA, M. The program solves for the threshold corresponding to PFA and M. A root finding technique is then employed to determine the SNR corresponding to this threshold and user-defined PD and M. The results are printed.

Description of Input

Inputs to the program consist of cards which either specify values (PARAMETER CARDS), activate the reading of tabularized values (TABLES), assign files (FILE NAME CARDS), process data (COMMAND CARDS), or specify a plot device (PLOT DEVICE CARDS). The basic format of a card is

CARD NAME = value units

where CARD NAME is an alphanumeric expression from Tables 2-6. The alphanumeric must begin in column 1, value is a floating point or integer number, and units is an alphanumeric.

Parameter cards, file names, and tables constitute the data upon which commands operate. If two cards with the same name specify different data, then the last entry overrides the other.

For the programmers convenience, FORTRAN variable names associated with file names or parameters may be located in the Tables 2 through 6. Since input and values stored represent the same physical quantity, it is convenient to refer to both in this paper by the same variable name. The convention adopted is to express the variable by the lower case letters and reserve upper case letters for constants.

Parameter Cards

Parameter cards are used to specify an axis length or assign a range of values to a parameter. These cards are shown in Table 2. For example,

```
NUMBER OF SAMPLES MINIMUM = 1.
NUMBER OF SAMPLES MAXIMUM = 8192.
NUMBER OF SAMPLES FACTOR  = 2.
```

implies that the program will process data for M=1,2,4,8,16,....,4096,8192.

Table Cards

A table card contains the values that are to be assigned to a variable. The last card that must appear in a table is an EOF card. This card terminates the reading of the table. Table cards exist for PD and PFA only. A list of the table cards appears in Table 4. For example,

PROBABILITY OF DETECTION TABLE
.5
.7
.99
EOF

This table assigns values of .5, .7, .99 to PD.

Files Cards

A file card allows for dynamic assignment of all mass storage files. This is accomplished by linking internal FORTRAN unit numbers to files during execution. The file card is shown in Table 4. Two of the three algorithms use files. They are

Display PD vs PFA : A file is used to store output.
Display SNR vs M : A file is used to store output.

For example,

OUTPUT FILE = PDFILE

directs the output of a program to a file called PDFILE.

Command Cards

Command cards are used to compute, plot, or terminate a run stream. Command cards are given in Table 5.

Plot Device Cards

Plot device cards direct the plot output to either a TEKTRONIX, FR80, or a CALCOMP plotter. The cards necessary for that operation are shown in Table 6.

Examples of Output

Example 1: Display PD vs PFA

The input deck for the first example appears in Table 7. This deck designates that PD vs. PFA data will be computed for M=1 and alpha=.5,1.0,1.5,...,9.5. The output is stored on a file called FILE1. The plot corresponding to the data is shown in figure 6. The second half of the run stream computes PD vs. PFA data for M=2 and alpha=0.,.4,.8,...,7.2. The output is stored in FILE2. The plot of the data appears in figure 7.

Example 2: Display SNR vs. M

The input deck for the second example appears in Table 8. The first half of the input deck designates that the SNR vs. M plots will be computed for a value PD=.5. The output is displayed in figure 1. The parameter cards specify that the axis will be scaled as follows: -19 DB (minimum), 13 DB (maximum), 2 DB (increment), and 5 inches long for the SNR axis and 6.86 inches long for the number of samples axis. It should be noted that the limits for the number of samples axis are predefined by the program to be 1 (minimum), 8192 (maximum), 2 (factor). The output is stored in a file called PDFIL1. The second half of the run stream computes SNR vs. M for a value PD=.9. The axis limits for SNR were changed to -17 DB (minimum), 15 DB (maximum), 2 DB (increment). Alpha curves were computed for alpha=0.,.4,.8,...,7.2. This output is stored in file PDFIL2. A plot of this data appears in figure 2.

Example 3: Print SNR

The input deck for the third example appears in Table 9. The output appears in Table 10.

TABLE 2. PARAMETER CARDS

INPUT CARDS	UNITS
SNR AXIS LENGTH = snraxs	IN
SAMPLE AXIS LENGTH = smpaxs	IN
PD AXIS LENGTH = pdaxs	IN
PFA AXIS LENGTH = pfaaxs	IN
SNR MINIMUM = snrmin	DB
SNR MAXIMUM = snrmax	DB
SNR INCREMENT = snrinc	DB
ALPHA MINIMUM = alpmin	
ALPHA MAXIMUM = alpmax	
ALPHA INCREMENT = alpinc	
NUMBER OF SAMPLES MINIMUM = smpmin	
NUMBER OF SAMPLES MAXIMUM = smpmax	
NUMBER OF SAMPLES FACTOR = smpfct	

TABLE 3. TABLE CARDS

INPUT CARDS	VARIABLE
PROBABILITY OF DETECTION TABLE	PD
PROBABILITY OF FALSE ALARM TABLE	PFA

TABLE 4. FILE CARDS

INPUT CARDS
OUTPUT FILE = name

TABLE 5. COMMAND CARDS

INPUT CARDS
RUN MAIN
COMPUTE PD VS PFA
COMPUTE SNR VS M
PLOT PD VS PFA
PLOT SNR VS M
END

TABLE 6. PLOT DEVICE CARDS

INPUT CARDS	OPTIONS
BAUD RATE = 960.	
PLOT DEVICE = device	FR80,TEKTRO,CALCOMP
RESET PLOT DEVICE	

TABLE 7. SAMPLE INPUT DECK FOR PD VS PFA

```
RUN MAIN
BAUD RATE = 960.
PLOT DEVICE = TEKTRO
RESET PLOT DEVICE
PD AXIS LENGTH = 6.86 IN
PFA AXIS LENGTH = 5. IN
OUTPUT FILE = FILE1
NUMBER OF SAMPLES MINIMUM = 1
ALPHA MINIMUM = .5
ALPHA MAXIMUM = 9.5
ALPHA INCREMENT = .5
COMPUTE PD VS PFA
PLOT PD VS PFA
OUTPUT FILE = FILE2
NUMBER OF SAMPLES MINIMUM = 2
ALPHA MINIMUM = 0.
ALPHA MAXIMUM = 7.2
ALPHA INCREMENT = .4
COMPUTE PD VS PFA
PLOT PD VS PFA
END
```

TABLE 8. SAMPLE INPUT DECK FOR SNR vs M

```
RUN MAIN
BAUD RATE = 960.
TEMPORARY FILE = FALSE
PLOT DEVICE = TEKTRO
RESET PLOT DEVICE
OUTPUT FILE = PDFIL1
SNR MINIMUM = -19. DB
SNR MAXIMUM = 13. DB
SNR INCREMENT = 2. DB
SNR AXIS LENGTH = 5. IN
SAMPLE AXIS LENGTH = 6.86 IN
PROBABILITY OF DETECTION TABLE
.5
EOF
COMPUTE SNR VS M
PLOT SNR VS M
OUTPUT FILE = PDFIL2
SNR MINIMUM = -17. DB
SNR MAXIMUM = 15. DB
SNR INCREMENT = 2. DB
PROBABILITY OF DETECTION TABLE
.9
EOF
COMPUTE SNR VS M
PLOT SNR VS M
END
```

TABLE 9. SAMPLE INPUT DECK FOR PRINTING SNR

```
RUN MAIN
PROBABILITY OF DETECTION TABLE
.5
.9
EOF
PROBABILITY OF FALSE ALARM TABLE
.1
.001
EOF
NUMBER OF SAMPLES MINIMUM = 1.
NUMBER OF SAMPLES MAXIMUM = 2048.
NUMBER OF SAMPLES FACTOR = 2.
PRINT SNR
END
```

TABLE 10. PRINT OUT OF SNR VS M

PD = 0.500 PFA = 0.100D+00

M	SNR
1	2.50
2	0.68
4	-1.08
8	-2.77
16	-4.41
32	-6.02
64	-7.59
128	-9.15
256	-10.69
512	-12.22
1024	-13.74
2048	-15.26

PD = 0.900 PFA = 0.100D+00

M	SNR
1	7.18
2	5.03
4	2.99
8	1.05
16	-0.79
32	-2.55
64	-4.24
128	-5.89
256	-7.50
512	-9.08
1024	-10.64
2048	-12.18

PD = 0.500 PFA = 0.100D-02

M	SNR
1	8.06
2	5.74
4	3.63
8	1.69
16	-0.14
32	-1.87
64	-3.54
128	-5.16
256	-6.75
512	-8.31
1024	-9.86
2048	-11.39

PD = 0.900 PFA = 0.100D-02

M	SNR
1	10.76
2	8.29
4	6.01
8	3.89
16	1.92
32	0.06
64	-1.72
128	-3.42
256	-5.07
512	-6.68
1024	-8.26
2048	-9.82

Listing of Program

This section contains a listing of three master programs and associated subroutines. Subroutines which read input and plot the output have been omitted. Table 11 contains a list of the subroutine names and a brief description of the pertinent subroutines.

TABLE 11. DESCRIPTION OF SUBROUTINES

NAME	DESCRIPTION
CMPDVA	MASTER PROGRAM FOR COMPUTING PD VS PFA
CMPSVS	MASTER PROGRAM FOR COMPUTING SNR VS M
PRTSNR	MASTER PROGRAM FOR COMPUTING AND PRINTING SNR
FFT	COMPUTES THE FAST FOURIER TRANSFORM OF A FUNCTION
RDC	COMPUTES AN APPROXIMATE S/N FOR A GIVEN PD, PFA, M (SEE REF 7)
FNPD	COMPUTES THE PROBABILITY OF DETECTION FOR A GIVEN M, S/N, AND THRESHOLD
FNPF	COMPUTES THE PROBABILITY OF FALSE ALARM FOR A GIVEN M AND THRESHOLD
FNF11	COMPUTES THE CONFLUENT HYPERGEOMETRIC FUNCTION
RICE	COMPUTES THE CHARACTERISTIC FUNCTION OF A RICE VARIATE
FNIPHI	COMPUTES THE INVERSE OF THE CUMULATIVE GAUSSIAN DISTRIBUTION
DIST	COMPUTES THE EXCEEDANCE DISTRIBUTION FUNCTION FOR A GIVEN M AND S/N

```
      SUBROUTINE FFT(N,X,Y)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION C(0:256),X(0:1023),Y(0:1023),L(0:9)
      DATA PI/3.14159265358979324D0/

      T=2.D0*PI/N
      J1=N/4
      DO 100 J=0,J1
      C(J)=DCOS(T*DFLOTJ(J))
 100  CONTINUE

      N1=N/4
      N2=N1+1
      N3=N2+1
      N4=N3+N1
      L2=JIDINT(1.4427D0*DLOG(DFLOTJ(N))+.5D0)
      DO 600 I1=1,L2
      I2=2**(L2-I1)
      I3=2D0*I2
      I4=N/I3

      DO 500 I5=1,I2
      I6=I4*(I5-1)+1
      IF( I6.LE.N2 ) GO TO 350
      V6=-C(N4-I6-1)
      V7=-C(I6-N1-1)
      GO TO 375
 350  V6=C(I6-1)
      V7=-C(N3-I6-1)
 375  L3=N-I3

      DO 400 I7=0,L3,I3
      I8=I7+I5
      I9=I8+I2
      V8=X(I8-1)-X(I9-1)
      V9=Y(I8-1)-Y(I9-1)
      X(I8-1)=X(I8-1)+X(I9-1)
      Y(I8-1)=Y(I8-1)+Y(I9-1)
      X(I9-1)=V6*V8-V7*V9
      Y(I9-1)=V6*V9+V7*V8
 400  CONTINUE
 500  CONTINUE
 600  CONTINUE


      I1=L2+1
      DO 700 I2=1,10
      L(I2-1)=1.D0
      IF( I2.GT.L2 ) GO TO 700
      L(I2-1)=2**(I1-I2)
 700  CONTINUE
```

```
      IC0=L(0)
      IC1=L(1)
      IC2=L(2)
      IC3=L(3)
      IC4=L(4)
      IC5=L(5)
      IC6=L(6)
      IC7=L(7)
      IC8=L(8)
      IC9=L(9)

      K=1
      DO 1900 I1=1,IC9
      DO 1800 I2=I1,IC8,IC9
      DO 1700 I3=I2,IC7,IC8
      DO 1600 I4=I3,IC6,IC7
      DO 1500 I5=I4,IC5,IC6
      DO 1400 I6=I5,IC4,IC5
      DO 1300 I7=I6,IC3,IC4
      DO 1200 I8=I7,IC2,IC3
      DO 1100 I9=I8,IC1,IC2
      DO 1000 I10=I9,IC0,IC1
      J=I10
      IF( K.GT.J ) GO TO 900
      A=X(K-1)
      X(K-1)=X(J-1)
      X(J-1)=A
      A=Y(K-1)
      Y(K-1)=Y(J-1)
      Y(J-1)=A
  900 K=K+1
 1000 CONTINUE
 1100 CONTINUE
 1200 CONTINUE
 1300 CONTINUE
 1400 CONTINUE
 1500 CONTINUE
 1600 CONTINUE
 1700 CONTINUE
 1800 CONTINUE
 1900 CONTINUE

      RETURN
      END
```

```
      SUBROUTINE FNPD(ALPHA,V,AM,AL,AD,ABS,PD)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA PI/3.14159265358979324D0/

      SNR=.5D0*ALPHA*ALPHA
      CALL FNF11(1.5D0,1.D0,SNR,F11)
      FAC=DSQRT(.5D0*PI)*EXP(-SNR)*F11
      AMUY=AM*FAC+ABS
      AM2=AM/2.D0
      VD=V*AD
      EXC=.5*AD*AMUY
      NS1=JIDINT(AL/AD)
      DO 100 NS=1,NS1
      XI=AD*NS
      CALL RICE(XI,SNR,FR,FI)
      A=DATAN2(FI,FR)
      FYI=DSIN(AM*A+ABS*XI)*(FR*FR+FI*FI)**AM2
      ADD=FYI*DCOS(VD*DFLOTJ(NS))/DFLOTJ(NS)
      EXC=EXC+ADD
100   CONTINUE
      PD=2.D0*EXC/PI

      RETURN
      END


      SUBROUTINE FNPF(V,AM,AL,AD,ABS,PF)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA PI/3.14159265358979324D0/

      FAC=DSQRT(.5D0*PI)
      AMUY=AM*FAC+ABS
      AM2=AM/2.D0
      VD=V*AD
      EXC=.5*AD*AMUY
      NS1=JIDINT(AL/AD)
      DO 100 NS=1,NS1
      XI=AD*NS
      X2=.5D0*XI*XI
      E=EXP(-X2)
      CALL FNF11(-.5D0,.5D0,X2,F11)
      FR=E*F11
      FI=E*FAC*XI
      A=DATAN2(FI,FR)
      FYI=DSIN(AM*A+ABS*XI)*(FR*FR+FI*FI)**AM2
      ADD=FYI*DCOS(VD*DFLOTJ(NS))/DFLOTJ(NS)
      EXC=EXC+ADD
100   CONTINUE
      PF=2.D0*EXC/PI

      RETURN
      END
```

```
      SUBROUTINE RDC(AM,PF,PD,ALPHA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

      A=DLOG(.62D0/PF)
      B=DLOG(PD/(1.D0-PD))
      FACT=6.2D0 + 4.54D0/DSQRT(AM+.44D0)
      SNRDB=-5.D0*DLOG10(AM) + DLOG10(A+.12D0*A*B+1.7D0*B)*FACT
      ALPHA=DSQRT(2.D0*10.D0**(.1D0*SNRDB))
      RETURN
      END


      SUBROUTINE FNF11(A,B,X,F11)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

      F11=1.D0
      T=1.D0
      DO 100 K=1,300
      U=K-1
      T=T*(A+U)*X/((B+U)*K)
      F11=F11+T
      IF( DABS(T).LE.DABS(F11)*1.D-18 ) GO TO 200
  100 CONTINUE
      PRINT 101
  101 FORMAT(2X,'300 TERMS IN FNF11')
  200 CONTINUE

      RETURN
      END


      SUBROUTINE FNIPHI(X,PHI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)

  100 Y=DMAX1(X,1.D-12)
      Y=DMIN1(Y,1.D0-1.D-12)
      D=X-.5D0
      IF( DABS(D).GT. .01D0 ) GO TO 250
      PHI=2.50662827463D0*D*(1.D0+D*D*1.04719755120D0)
      GO TO 300
  250 PHI=Y
      IF( Y.GT. .5D0 ) PHI=.5D0-(Y-.5D0)
      PHI=DSQRT(-2.D0*DLOG(PHI))
      T=1.D0+PHI*(1.432788D0+PHI*(.189269D0+PHI*.001308D0))
      PHI=PHI-(2.515517D0+PHI*(.802853D0+PHI*.010328D0))/T
      IF( Y.LT. .5D0 ) PHI=-PHI
  300 RETURN
      END
```

```
      SUBROUTINE DIST(AM,ALPHA,MF,X,Y)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION X(0:1023), Y(0:1023)
      COMMON /PDVPF/AL,AD,ABS
      DATA PI/3.14159265358979324D0/


      SNR=.5D0*ALPHA*ALPHA
      CALL FNF11(1.5D0,1.D0,SNR,F11)
      AMU=DSQRT(.5D0*PI)*DEXP(-SNR)*F11
      AMUS=AM*AMU+ABS
      AM2=AM/2.D0

      DO 100 I=0,1023
      X(I)=0.D0
      Y(I)=0.D0
  100 CONTINUE

      X(0)=.5D0*AMUS*AD
      NS1=JIDINT(AL/AD)
      DO 1000 NS=1,NS1
      XI=AD*NS
      CALL RICE(XI,SNR,U,V)
      T=DATAN2(V,U)
      FI=DSIN(AM*T+ABS*XI)*(U*U+V*V)**AM2
      MS=JMOD(NS,MF)
      X(MS)=X(MS)+FI/NS
 1000 CONTINUE

      CALL FFT(MF,X,Y)

      FAC=2.D0/PI
      KS1=MF/2.D0
      DO 2000 KS=0,KS1
      T=X(KS)*FAC
      X(KS)=1.D0-T
      Y(KS)=T
 2000 CONTINUE

      RETURN
      END
```

```
      SUBROUTINE RICE(X,SNR,FR,FI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DATA PI/3.14159265358979324D0/


      X2=.5D0*X*X
      E=DEXP(-X2-SNR)
      CALL FNF11(-.5D0,.5D0,X2,F11)
      AOR=E*F11
      AOI=E*DSQRT(.5D0*PI)*X
      CALL FNF11(-1.5D0,.5D0,X2,F11)
      ANR=E*SNR*F11
      ANI=SNR*(1.5D0-X2)*AOI
      FR=AOR+ANR
      FI=AOI+ANI
      BR=DMAX1(DABS(AOR),DABS(FR))
      BI=DMAX1(DABS(AOI),DABS(FI))
      T=.5D0+X2

      SNR2=SNR**2
      DO 100 N=2,200
      FO=N**2
      F1=SNR*(N+N-T)/FO
      F2=SNR2*(N-.5D0)/((N-1)*FO)
      R=F1*ANR-F2*AOR
      V=F1*ANI-F2*AOI
      AOR=ANR
      AOI=ANI
      ANR=R
      ANI=V
      FR=FR+R
      FI=FI+V
      BR=DMAX1(BR,DABS(FR))
      BI=DMAX1(BI,DABS(FI))
      IF( DABS(V).LE.5.D-19*DABS(FI) .AND. DABS(R).LE.5.D-19*DABS(FR))
    1 GO TO 200
100   CONTINUE
       PRINT 101
101   FORMAT(2X,'200 TERMS IN RICE')
200   DR=18.-DLOG10(DABS(BR/FR))
      DI=18.-DLOG10(DABS(BI/FI))
      RETURN
      END
```

```
      SUBROUTINE CMPDVA
      PARAMETER MF=2**10
      PARAMETER PBDNUM=18
      DOUBLE PRECISION AL,AD,ABS,BSA,AM,ALPHA,ALFA,X(0:1023),Y(0:1023)
      PARAMETER  (NUMFIL=30)
      CHARACTER*6  FILES(NUMFIL)
      COMMON  /FILEC/FILES
      CHARACTER*6  PBDNAM
      EQUIVALENCE
     1  (PBDNAM,FILES(18))
      PARAMETER  (NUMPAR=200)
      COMMON  /PARAMC/PARAMS(NUMPAR)
      EQUIVALENCE
     1  (SMPMIN,PARAMS(187)),
     1  (SNMIN,PARAMS(184)), (SNMAX,PARAMS(185)), (SNDEL,PARAMS(186))
      COMMON/PDVPF/AL,AD,ABS
      DOUBLE PRECISION PI
      DATA PI/3.14159265358979324D0/

C
C       OPEN THE FILE
      CALL OPNFIL(PBDNUM,PBDNAM)
C
C       COMPUTE THE NUMBER OF SNR CURVES
C
      NSN=(SNMAX-SNMIN)/SNDEL + 1

C
C       STORE HEADER INFO
      WRITE(PBDNUM) SMPMIN,SNMIN,SNMAX,SNDEL,NSN

      AM = SMPMIN
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
C
C       COMPUTE SNR VS PFA
      ALFA=0.D0
      CALL DIST(AM,ALFA,MF,X,Y)
C
C       STORE THE SNR VS PD
      WRITE(PBDNUM) (Y(I),I=0,512)

      DO 1000 ISN=1,NSN
      SNR= SNMIN + SNDEL*(ISN-1)

      ALPHA = SNR
      CALL DIST(AM,ALPHA,MF,X,Y)
C
C       STORE THE SNR VS PD
      WRITE(PBDNUM) (Y(I),I=0,512)

 1000 CONTINUE

 2000 CONTINUE

      RETURN
      END
```

```
      SUBROUTINE PRTSNR
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION PFA(10),PD(10),V(14,8),SNR(14,8)
      DATA PI/3.14159265358979324D0/
      REAL SMPMIN,SMPMAX,SMPFCT,PARAMS
      PARAMETER NUMPAR=200
      COMMON/PARAMC/PARAMS(NUMPAR)
      EQUIVALENCE
     1 (SMPMIN,PARAMS(187)),(SMPMAX,PARAMS(188)), (SMPFCT,PARAMS
      COMMON/PDPF/NPD,NPFA,PD,PFA


      MMAX=ALOG10(SMPMAX/SMPMIN)/ALOG10(SMPFCT) + 1



      F1=DSQRT(.5D0*PI)
      F2=DSQRT(2.D0-.5D0*PI)
      DO 1000 IM=1,MMAX
      AM=SMPMIN*SMPFCT**(IM-1)
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
      AMU=F1*AM
      SIG=F2*DSQRT(AM)
      DO 900 IPF=1,NPFA
      PF=PFA(IPF)
      IF( AM.GT. 1.D0 ) GO TO 250
      VN=DSQRT(-2.*DLOG(PF))
      GO TO 750
  250 CALL FNIPHI(PF,YF)
      V1=AMU-SIG*YF+ABS
      IF( IPF.GT.1 ) V1=DMAX1(V1,VN)
      V2=V1+.5D0
      IF( V1.NE.VN ) GO TO 300
      P1=PN
      GO TO 325
  300 CALL FNPF(V1,AM,AL,AD,ABS,P1)
  325 CALL FNPF(V2,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PF).LT.DABS(P2-PF) ) GO TO 350
      VO=V1
      PO=P1
      VN=V2
      PN=P2
      GO TO 400
  350 VO=V2
      PO=P2
      VN=V1
      PN=P1
  400 CALL FNIPHI(PO,YO)
      GO TO 550
  500 CALL FNPF(VN,AM,AL,AD,ABS,PN)
  550 CALL FNIPHI(PN,YN)
      IF( DABS(PN-PF).LE.1D-9*PF ) GO TO 750
      T=(VO*(YN-YF)+VN*(YF-YO))/(YN-YO)
```

```
      VO=VN
      YO=YN
      VN=T
      GO TO 500
 750  V(IM,IPF)=VN
 900  CONTINUE
1000  CONTINUE

      DO 4000 IPD=1,NPD

      CALL FNIPHI(PD(IPD),YD)
      DO 3000 IM=1,MMAX
      AM=SMPMIN*SMPFCT**(IM-1)
      AL = DMIN1(9.DO,17.DO/DSQRT(AM))
      AD = .12DO/DSQRT(AM)
      BSA = -DSQRT(PI/2.DO)*AM + 6.DO*DSQRT(AM)
      ABS = DMIN1(0.DO,BSA)
      DO 2900 IPF=1,NPFA
      PF=PFA(IPF)
      CALL RDC(AM,PF,PD(IPD),A1)
      A2=A1*1.01DO
      VV=V(IM,IPF)
      CALL FNPD(A1,VV,AM,AL,AD,ABS,P1)
      CALL FNPD(A2,VV,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PD(IPD)).LT.DABS(P2-PD(IPD)) ) GO TO 2350
      AO=A1
      PO=P1
      AN=A2
      PN=P2
      GO TO 2400
2350  AO=A2
      PO=P2
      AN=A1
      PN=P1
2400  CALL FNIPHI(PO,YO)
      GO TO 2550
2500  CALL FNPD(AN,VV,AM,AL,AD,ABS,PN)
2550  CALL FNIPHI(PN,YN)
      IF( DABS(PN-PD(IPD)).LE.1D-6*PD(IPD) ) GO TO 2750
      T=(AO*(YN-YD)+AN*(YD-YO))/(YN-YO)
      AO=AN
      YO=YN
      AN=T
      GO TO 2500
2750  SNR(IM,IPF)=10.*DLOG10(.5DO*AN*AN)
2900  CONTINUE
3000  CONTINUE
```

```
      DO 3200 IPF=1,NPFA
      PRINT 3001
 3001 FORMAT(2(/))
      PRINT 3011, PD(IPD),PFA(IPF)
 3011 FORMAT(2X,'PD =',F10.3,5X,'PFA =',D10.3)
      DO 3100 IM=1,MMAX
      M=SMPMIN*SMPFCT**(IM-1)
      PRINT 3021, M,SNR(IM,IPF)
 3021 FORMAT(2X,'M =',I5,5X,'SNR =',F7.2)
 3100 CONTINUE
 3200 CONTINUE

 4000 CONTINUE


      RETURN
      END
```

```
      SUBROUTINE CMPSVS
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER MMAX=14
      PARAMETER NUMFIL=30, PBDNUM=18
      CHARACTER*6 FILES(NUMFIL)
      COMMON/FILEC/FILES
      CHARACTER*6 PBDNAM
      EQUIVALENCE (PBDNAM,FILES(18))
      DIMENSION PFA(10),PD(10),V(14,8),ALPHA(14,8)
      DIMENSION THRS(14,8)
      DATA PI/3.14159265358979324D0/
      COMMON/PDPF/NPD,NPFA,PD,PFA


      CALL OPNFIL(PBDNUM,PBDNAM)
```

```
      F1=DSQRT(.5D0*PI)
      F2=DSQRT(2.D0-.5D0*PI)
      DO 1000 IM=1,MMAX
      AM=2.**(IM-1)
      AL = DMIN1(9.D0,17.D0/DSQRT(AM))
      AD = .12D0/DSQRT(AM)
      BSA = -DSQRT(PI/2.D0)*AM + 6.D0*DSQRT(AM)
      ABS = DMIN1(0.D0,BSA)
      AMU=F1*AM
      SIG=F2*DSQRT(AM)
      DO 900 IPF=1,8
      PF=10.**(-DFLOTJ(IPF))
      IF( AM.GT. 1.D0 ) GO TO 250
      VN=DSQRT(-2.*DLOG(PF))
      GO TO 750
  250 CALL FNIPHI(PF,YF)
      V1=AMU-SIG*YF+ABS
      IF( IPF.GT.1 ) V1=DMAX1(V1,VN)
      V2=V1+.5D0
      IF( V1.NE.VN ) GO TO 300
      P1=PN
      GO TO 325
  300 CALL FNPF(V1,AM,AL,AD,ABS,P1)
  325 CALL FNPF(V2,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PF).LT.DABS(P2-PF) ) GO TO 350
      V0=V1
      P0=P1
      VN=V2
      PN=P2
      GO TO 400
  350 V0=V2
      P0=P2
      VN=V1
      PN=P1
  400 CALL FNIPHI(P0,Y0)
      GO TO 550
  500 CALL FNPF(VN,AM,AL,AD,ABS,PN)
  550 CALL FNIPHI(PN,YN)
      IF( DABS(PN-PF).LE.1D-9*PF ) GO TO 750
      T=(V0*(YN-YF)+VN*(YF-Y0))/(YN-Y0)
      V0=VN
      Y0=YN
      VN=T
      GO TO 500
  750 V(IM,IPF)=VN
      THRS(IM,IPF)=(VN-ABS)/AM
  900 CONTINUE
 1000 CONTINUE
```

```
      WRITE(PBDNUM)    NPD,(PD(I),I=1,10)
      DO 4000 IPD=1,NPD

      CALL FNIPHI(PD(IPD),YD)
      DO 3000 IM=1,MMAX
      AM=2.DO**(IM-1)
      AL = DMIN1(9.DO,17.DO/DSQRT(AM))
      AD = .12DO/DSQRT(AM)
      BSA = -DSQRT(PI/2.DO)*AM + 6.DO*DSQRT(AM)
      ABS = DMIN1(0.DO,BSA)
      DO 2900 IPF=1,8
      PF=10.DO**(-DFLOTJ(IPF))
      CALL RDC(AM,PF,PD(IPD),A1)
      A2=A1*1.01DO
      VV=V(IM,IPF)
      CALL FNPD(A1,VV,AM,AL,AD,ABS,P1)
      CALL FNPD(A2,VV,AM,AL,AD,ABS,P2)
      IF( DABS(P1-PD(IPD)).LT.DABS(P2-PD(IPD)) ) GO TO 2350
      AO=A1
      PO=P1
      AN=A2
      PN=P2
      GO TO 2400
2350  AO=A2
      PO=P2
      AN=A1
      PN=P1
2400  CALL FNIPHI(PO,YO)
      GO TO 2550
2500  CALL FNPD(AN,VV,AM,AL,AD,ABS,PN)
2550  CALL FNIPHI(PN,YN)
      IF( DABS(PN-PD(IPD)).LE.1D-6*PD(IPD) ) GO TO 2750
      T=(AO*(YN-YD)+AN*(YD-YO))/(YN-YO)
      AO=AN
      YO=YN
      AN=T
      GO TO 2500
2750  ALPHA(IM,IPF)=AN
2900  CONTINUE
3000  CONTINUE

      WRITE(PBDNUM) ((ALPHA(IM,IPF),IPF=1,8),IM=1,MMAX)

4000  CONTINUE

      RETURN
      END
$
```

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